Solution (#1559) Recall from #1558 that

$$\int_{-\infty}^{\infty} e^{-x^2} \cos 2ax \, \mathrm{d}x = \sqrt{\pi} e^{-a^2}.$$

As

$$\begin{aligned} \cos^2 x &= \frac{1}{2}(1+\cos 2x); \\ \sin^2 x &= \frac{1}{2}(1-\cos 2x); \\ \cos^3 x &= \frac{1}{4}(\cos 3x + 3\cos x); \\ \sin^4 x &= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x, \end{aligned}$$

then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2} \cos^2 x \, \mathrm{d}x &= \frac{\sqrt{\pi}}{2}\left(1+e^{-1}\right). \\ \int_{-\infty}^{\infty} e^{-x^2} \sin^2 x \, \mathrm{d}x &= \frac{\sqrt{\pi}}{2}\left(1-e^{-1}\right). \\ \int_{-\infty}^{\infty} e^{-x^2} \cos^3 x \, \mathrm{d}x &= \frac{\sqrt{\pi}}{4}\left(3e^{-1/4}+e^{-9/4}\right). \\ \int_{-\infty}^{\infty} e^{-x^2} \sin^4 x \, \mathrm{d}x &= \frac{\sqrt{\pi}}{8}\left(3-4e^{-1}+e^{-4}\right). \end{aligned}$$