Solution (#1561) Let $a > 0$ and define

$$I(a) = \int_{-\infty}^{\infty} \frac{\cos ax}{x^4+1}\,\mathrm{d}x.$$

If we apply IBP we find

$$I(a) = \left[\frac{\sin ax}{a}\frac{1}{x^4+1}\right]_{-\infty}^{\infty} + \frac{1}{a}\int_{-\infty}^{\infty} \frac{4x^3 \sin ax}{(x^4+1)^2}\,\mathrm{d}x = \frac{4}{a}\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{(x^4+1)^2}\,\mathrm{d}x.$$

Differentiating the identity

$$aI(a) = 4\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{(x^4+1)^2}\,\mathrm{d}x$$

we obtain

$$aI'(a) + I(a) = 4\int_{-\infty}^{\infty} \frac{x^4 \cos ax}{(x^4+1)^2}\,\mathrm{d}x = 4I(a) - 4\int_{-\infty}^{\infty} \frac{\cos ax}{(x^4+1)^2}\,\mathrm{d}x$$

and so

$$aI'(a) - 3I(a) = -4\int_{-\infty}^{\infty} \frac{\cos ax}{(x^4+1)^2}\,\mathrm{d}x.$$

Differentiating this three times we find

$$aI''''(a) + 3I'''(a) - 3I'''(a) = -4\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{(x^4+1)^2}\,\mathrm{d}x.$$

Finally if we simplify and IBP then we have

$$aI''''(a) = \left[\frac{\sin ax}{x^4+1}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{a\cos ax}{x^4+1}\,\mathrm{d}x = -aI(a).$$

Hence $I'''(a) = -I(a)$.

This DE has auxiliary equation $m^4 = -1$ which has roots $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$. So by Theorem 6.28 the general solution is

$$I(a) = e^{a/\sqrt{2}}\left(A\cos\frac{a}{\sqrt{2}} + B\sin\frac{a}{\sqrt{2}}\right) + e^{-a/\sqrt{2}}\left(C\cos\frac{a}{\sqrt{2}} + D\sin\frac{a}{\sqrt{2}}\right).$$

As $I(a)$ is bounded by $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x$ then we see $A = B = 0$. We also know from #1433 that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}} \qquad \text{and also} \qquad \int_{-\infty}^{\infty} \frac{x}{x^4+1}\,\mathrm{d}x = 0$$

as the second integrand is odd. So

$$C = \frac{\pi}{\sqrt{2}} \qquad \text{and also} \qquad -\frac{1}{\sqrt{2}}C + \frac{1}{\sqrt{2}}D = 0,$$

so that $C = D = \pi/\sqrt{2}$ and

$$I(a) = \frac{\pi}{\sqrt{2}}e^{-a/\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right).$$