Solution (#1562) For $a \geqslant 0$ set

$$I(a)=\int_0^\infty \frac{e^{-ax^2}}{x^2+1}\,\mathrm{d}x.$$

Differentiating under the integral sign we find

$$I'(a)=\int_0^\infty \frac{-x^2e^{-ax^2}}{x^2+1}\,\mathrm{d}x=-\int_0^\infty e^{-ax^2}\,\mathrm{d}x+\int_0^\infty \frac{e^{-ax^2}}{x^2+1}\,\mathrm{d}x=-\frac{1}{2}\sqrt{\frac{\pi}{a}}+I(a),$$

by #1401. So

$$I'(a)-I(a)=-\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{a}}.$$

This DE can be solved using integrating factors. Here the integrating factor is e^{-a} and we see

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(I(a)e^{-a}\right)=I'(a)e^{-a}-I(a)e^{-a}=-\frac{\sqrt{\pi}}{2}\frac{e^{-a}}{\sqrt{a}},$$

and hence

$$I(a)e^{-a}=I(0)-\frac{\sqrt{\pi}}{2}\int_0^a \frac{e^{-x}}{\sqrt{x}}\,\mathrm{d}x.$$

Now $I(0)=\pi/2$ and further making the substitution $x=u^2$ we see

$$\begin{aligned}
I(a)e^{-a} &= \frac{\pi}{2}-\frac{\sqrt{\pi}}{2}\int_0^{\sqrt{a}}\frac{e^{-u^2}}{u}\,2u\,\mathrm{d}u\\
&= \frac{\pi}{2}-\sqrt{\pi}\int_0^{\sqrt{a}} e^{-u^2}\,\mathrm{d}u\\
&= \frac{\pi}{2}-\frac{\pi}{2}\operatorname{erf}\sqrt{a},
\end{aligned}$$

finally giving

$$I(a)=\frac{\pi e^a}{2}\left(1-\operatorname{erf}\left(\sqrt{a}\right)\right).$$

The desired integral equals

$$I(1)=\frac{\pi e}{2}\left(1-\operatorname{erf}(1)\right).$$