**Solution** (#1563) Let $X, Y$ be independent normally distributed random variables with mean 0 and variance 1. Let $Z = X/Y$ and let $\Phi(x)$ denote the cdf of a $N(0,1)$ distribution – that is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,\mathrm{d}t.$$

Then the cumulative distribution function of $Z$ satisfies

$$F_Z(z) = P\left(\frac{X}{Y} \leqslant z\right) = P\left(X \leqslant zY \quad \text{and} \quad Y \geqslant 0\right) + P\left(X \geqslant zY \quad \text{and} \quad Y < 0\right).$$

Now

$$\begin{aligned} P\left(X \leqslant zY \quad \text{and} \quad Y \geqslant 0\right) &= \int_{y=0}^{\infty} \Phi(zy) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y; \\ P\left(X \geqslant zY \quad \text{and} \quad Y < 0\right) &= \int_{y=-\infty}^{0} (1-\Phi(zy)) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y, \end{aligned}$$

giving

$$F_Z(z) = \int_{y=0}^{\infty} \Phi(zy) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y + \int_{y=-\infty}^{0} (1-\Phi(zy)) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y.$$

Differentiating under the integral sign, we find

$$\begin{aligned} f_Z(z) &= \int_{y=0}^{\infty} y\Phi'(zy) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y - \int_{y=-\infty}^{0} y\Phi'(zy) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\ &= \int_{y=0}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y - \int_{y=-\infty}^{0} y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\ &= \frac{1}{2\pi} \int_{y=0}^{\infty} y e^{-\frac{1}{2}(z^2+1)y^2}\,\mathrm{d}y - \frac{1}{2\pi} \int_{y=-\infty}^{0} y e^{-\frac{1}{2}(z^2+1)y^2}\,\mathrm{d}y \\ &= \frac{1}{2\pi} \left[\frac{e^{-\frac{1}{2}(z^2+1)y^2}}{-(1+z^2)}\right]_0^{\infty} - \frac{1}{2\pi} \left[\frac{e^{-\frac{1}{2}(z^2+1)y^2}}{-(1+z^2)}\right]_0^{\infty} \\ &= \frac{1}{2\pi} \left(0 + \frac{1}{1+z^2} - 0 + \frac{1}{1+z^2}\right) \\ &= \frac{1}{\pi(1+z^2)}, \end{aligned}$$

which is the same as the pdf of the Cauchy distribution.