Solution (#1564) For $0 < k < 1$ define

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}\,\mathrm{d}x, \qquad K(k) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}}.$$

Differentiating under the integral sign we find

$$\begin{aligned} E'(k) &= \int_0^{\pi/2} \frac{\frac{1}{2}(-2k\sin^2 x)\,\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} \\ &= -\frac{1}{k}\int_0^{\pi/2} \frac{\left(k^2\sin^2 x - 1 + 1\right)\,\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} \\ &= -\frac{1}{k}\left(-E(k)+K(k)\right) \\ &= \frac{E(k)-K(k)}{k}. \end{aligned}$$

Similarly

$$\begin{aligned} K'(k) &= \int_0^{\pi/2} \frac{-\frac{1}{2}(-2k\sin^2 x)\,\mathrm{d}x}{(1-k^2\sin^2 x)^{3/2}} \\ &= \frac{1}{k}\int_0^{\pi/2} \frac{\left(k^2\sin^2 x - 1 + 1\right)\,\mathrm{d}x}{(1-k^2\sin^2 x)^{3/2}} \\ &= \frac{1}{k}\int_0^{\pi/2} \left[-\left(1-k^2\sin^2 x\right)^{-1/2} + (1-k^2\sin^2 x)^{-3/2}\right]\mathrm{d}x \\ &= \frac{1}{k}\left[-K(k) + \frac{E(k)}{1-k^2}\right] \qquad \text{[by \#1548]} \\ &= \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}. \end{aligned}$$

Now by #1548 we have

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{3/2}} = \frac{E(k)}{1-k^2}.$$

Differentiating under the integral sign we find

$$\int_0^{\pi/2} \frac{-\frac{3}{2}(-2k\sin^2 x)\,\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{5/2}} = \frac{E'(k)}{1-k^2} + \frac{2kE(k)}{(1-k^2)^2}.$$

This rearranges to

$$\frac{3}{k}\int_0^{\pi/2} \frac{(k^2\sin^2 x - 1 + 1)\,\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{5/2}} = \frac{(E(k)-K(k))}{k(1-k^2)} + \frac{2kE(k)}{(1-k^2)^2}$$

and so

$$\frac{3}{k}\left[-\int_0^{\pi/2} \frac{\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{3/2}} + \int_0^{\pi/2} \frac{\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{5/2}}\right] = \frac{(E(k)-K(k))}{k(1-k^2)} + \frac{2kE(k)}{(1-k^2)^2}.$$

Using #1548 again we have

$$\begin{aligned} \int_0^{\pi/2} \frac{\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{5/2}} &= \frac{1}{3}\left\{\frac{E(k)-K(k)}{1-k^2} + \frac{2k^2E(k)}{(1-k^2)^2} + \frac{3E(k)}{1-k^2}\right\} \\ &= \frac{1}{3}\left\{\frac{(4-2k^2)E(k)}{\left(1-k^2\right)^2} - \frac{K(k)}{1-k^2}\right\}. \end{aligned}$$

and so finally

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{\left(1-k^2\sin^2 x\right)^{5/2}} = \frac{1}{3}\left\{\frac{(4-2k^2)E(k)}{\left(1-k^2\right)^2} - \frac{K(k)}{1-k^2}\right\}.$$