Solution (#1566) (i) Let n be a positive integer. Then

$$\int_0^1 \frac{1-x^n}{1-x}\,\mathrm{d}x = \int_0^1 \left(1+x+x^2+\cdots+x^{n-1}\right)\,\mathrm{d}x = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = H_n.$$

(ii) So for $\alpha > -1$ we define

$$H_\alpha = \int_0^1 \frac{1-x^\alpha}{1-x}\,\mathrm{d}x.$$

If $\alpha > 0$ then

$$H_\alpha - H_{\alpha-1} = \left(\int_0^1 \frac{1-x^\alpha}{1-x}\,\mathrm{d}x\right) - \left(\int_0^1 \frac{1-x^{\alpha-1}}{1-x}\,\mathrm{d}x\right) = \int_0^1 \frac{x^{\alpha-1}-x^\alpha}{1-x}\,\mathrm{d}x = \int_0^1 x^{\alpha-1}\,\mathrm{d}x = \frac{1}{\alpha}.$$

Further for $0<x<1$ we have

$$\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots.$$

Let N be a positive integer. Then

$$\begin{aligned}
&\int_0^1 (1-x^\alpha)(1+x+x^2+\cdots+x^N)\,\mathrm{d}x \\
=\ &\left(1+\frac{1}{2}+\cdots+\frac{1}{N+1}-\frac{1}{\alpha+1}-\frac{1}{\alpha+2}-\cdots-\frac{1}{\alpha+N}\right) \\
=\ &\frac{\alpha}{1(\alpha+1)}+\frac{\alpha}{2(\alpha+2)}+\cdots+\frac{\alpha}{N(\alpha+N)} \\
=\ &\sum_{k=1}^{N}\frac{\alpha}{k(k+\alpha)}.
\end{aligned}$$

So

$$\left|\int_0^1 \frac{1-x^\alpha}{1-x}\,\mathrm{d}x - \sum_{k=1}^{N}\frac{\alpha}{k(k+\alpha)}\right| = \int_0^1 (1-x^\alpha)(x^{N+1}+x^{N+2}+\cdots)\,\mathrm{d}x = \int_0^1 \frac{(x-x^{\alpha+1})x^N}{1-x}\,\mathrm{d}x.$$

The function $(x-x^{\alpha+1})/(1-x)$ is bounded on the interval $(0,1)$, say by a positive number M, and so the above integral is less than $M/(n+1)$ which converges to zero. So letting N tend to infinity we see

$$\int_0^1 \frac{1-x^\alpha}{1-x}\,\mathrm{d}x = \sum_{k=1}^{\infty}\frac{\alpha}{k(k+\alpha)}.$$