Solution (#1567) Let n be a positive integer. Note by #1566(ii) that

$$\int_n^{n+1} H_x \,\mathrm{d}x = \int_n^{n+1} \left(H_{x-1} + \frac{1}{x} \right) \mathrm{d}x = \int_{n-1}^{n} H_x \,\mathrm{d}x + \ln\left(\frac{n+1}{n}\right) \mathrm{d}x.$$

Using the above we see

$$\int_{k-1}^{k} H_x \,\mathrm{d}x = \int_0^1 H_x \,\mathrm{d}x + \ln\frac{2}{1} + \ln\frac{3}{2} + \cdots + \ln\frac{k}{k-1} = \int_0^1 H_x \,\mathrm{d}x + \ln k.$$

Hence

$$\begin{aligned} \int_0^n H_x \,\mathrm{d}x &= \int_0^1 H_x \,\mathrm{d}x + \int_1^2 H_x \,\mathrm{d}x + \int_2^3 H_x \,\mathrm{d}x + \cdots + \int_{n-1}^n H_x \,\mathrm{d}x \\ &= \int_0^1 H_x \,\mathrm{d}x + \left(\int_0^1 H_x \,\mathrm{d}x + \ln 2\right) + \left(\int_0^1 H_x \,\mathrm{d}x + \ln 3\right) + \cdots \left(\int_0^1 H_x \,\mathrm{d}x + \ln n\right) \\ &= n\int_0^1 H_x \,\mathrm{d}x + \ln n!. \end{aligned}$$

Now we showed in #1566(ii) that

$$H_x = \sum_{k=1}^{\infty} \frac{x}{k(k+x)}.$$

Note, for a positive integer N, that

$$\begin{aligned} \int_0^1 \sum_{k=1}^{N} \frac{x}{k(k+x)} \,\mathrm{d}x &= \sum_{k=1}^{N} \int_0^1 \left(\frac{1}{k} - \frac{1}{k+x}\right) \mathrm{d}x \\ &= \sum_{k=1}^{N} \left(\frac{1}{k} - \ln\left(\frac{k+1}{k}\right)\right) \\ &= H_N - \ln(N+1) \end{aligned}$$

and this converges to γ as N tends to infinity. Finally we note

$$\left| \int_0^1 H_x \,\mathrm{d}x - \int_0^1 \sum_{k=1}^{N} \frac{x}{k(k+x)} \,\mathrm{d}x \right| = \int_0^1 \sum_{k=N+1}^{\infty} \frac{x}{k(k+x)} \,\mathrm{d}x \leqslant \int_0^1 \sum_{k=N+1}^{\infty} \frac{x}{k^2} \,\mathrm{d}x = \frac{1}{2} \sum_{k=N+1}^{\infty} \frac{1}{k^2}$$

which tends to zero as N tends to infinity. Thus we've shown

$$\int_0^1 H_x \,\mathrm{d}x = \lim_{N\to\infty} \left(H_N - \ln(N+1)\right) = \gamma$$

and from our earlier calculations can conclude that

$$\int_0^n H_x \,\mathrm{d}x = n\gamma + \ln n!.$$