Solution (#1570) By definition for $a > 0$ we have

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, \mathrm{d}t.$$

So if we differentiate under the integral sign we see

$$\Gamma'(a) = \frac{\mathrm{d}}{\mathrm{d}a} \int_0^\infty t^{a-1} e^{-t} \, \mathrm{d}t = \int_0^\infty \frac{\partial}{\partial a} \left(t^{a-1} e^{-t} \right) \, \mathrm{d}t = \int_0^\infty t^{a-1} e^{-t} \ln t \, \mathrm{d}t$$

as the derivative of $k^x = e^{x \ln k}$ with respect to x is $k^x \ln k$. Hence

$$\Gamma'(1) = \int_0^\infty e^{-t} \ln t \, \mathrm{d}t$$

and as $\Gamma(1) = 1$ and using #1569(iii) we have

$$\int_0^\infty e^{-t} \ln t \, \mathrm{d}t = \frac{\Gamma'(1)}{\Gamma(1)} = \psi(1) = H_0 - \gamma = -\gamma.$$

For the second integral we set $x = -\ln t$ to find

$$-\gamma = \int_0^\infty e^{-x} \ln x \, \mathrm{d}x = \int_1^0 e^{\ln t} \ln(-\ln t) \left(-\frac{\mathrm{d}t}{t} \right) = \int_0^1 \ln(-\ln t) \, \mathrm{d}t.$$