Solution (#1573) The moment generating function $M_X(t)$ of a continuous random variable with pdf f_X equals

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,\mathrm{d}x.$$

(i) uniform distribution on $a \leqslant x \leqslant b$: if $t \neq 0$ then

$$M(t) = \frac{1}{b-a}\int_a^b e^{tx}\,\mathrm{d}x = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{e^{bt}-e^{at}}{(b-a)t}.$$

At $t = 0$ this has a limit of 1 which agrees with the definition of $M(0)$ in any case.
(ii) exponential distribution with parameter $\lambda > 0$: for $t < \lambda$ we have

$$M(t) = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,\mathrm{d}x = \lambda\int_0^{\infty} e^{-(\lambda-t)x}\,\mathrm{d}x = \frac{\lambda}{\lambda-t}.$$

(iii) gamma distribution with parameters $a > 0, \lambda > 0$: for $t < \lambda$ we have

$$\begin{aligned}
M(t) &= \int_0^{\infty} e^{tx}\frac{1}{\Gamma(a)}\lambda^a x^{a-1}e^{-\lambda x}\,\mathrm{d}x \\
&= \frac{\lambda^a}{\Gamma(a)}\int_0^{\infty} x^{a-1}e^{-(\lambda-t)x}\,\mathrm{d}x \\
&= \frac{\lambda^a}{\Gamma(a)}\int_0^{\infty}\left(\frac{u}{\lambda-t}\right)^{a-1}e^{-u}\,\frac{\mathrm{d}u}{\lambda-t} \qquad [u=(\lambda-t)x] \\
&= \frac{\lambda^a}{\Gamma(a)}\times\frac{\Gamma(a)}{(\lambda-t)^a} \\
&= \left(\frac{\lambda}{\lambda-t}\right)^a.
\end{aligned}$$

Note unsurprisingly that this agrees with (ii) when $a = 1$.
(iv) Cauchy distribution: we have

$$M(t) = \int_{-\infty}^{\infty}\frac{e^{tx}\,\mathrm{d}x}{\pi(1+x^2)}.$$

If $t > 0$ then this integral does not converge on $(0,\infty)$ and if $t < 0$ the integral does not converge on $(-\infty, 0)$. So the moment generating function is only defined at $t = 0$ where $M(0) = 1$.