Solution (#495) Let S be a non-empty subset of $\mathbb{R}^2$ which is closed under addition and scalar multiplication. It is hopefully clear that each of $\{\mathbf{0}\}$, a line through the origin and $\mathbb{R}^2$ are closed under addition and scalar multiplication.

As S is non-empty then it contains at least one vector $\mathbf{u}$ and as it is closed under scalar multiplication then it contains $\mathbf{0} = 0\mathbf{u}$. If $S = \{\mathbf{0}\}$ then we are done.

Otherwise S contains a non-zero vector $\mathbf{v}$. Again as S is closed under scalar multiplication then S contains each vector $\lambda\mathbf{v}$ where λ is a real number. The vectors $\lambda\mathbf{v}$ comprise a line through the origin and if S is this line then we are again done.

Otherwise S contains a vector $\mathbf{w}$ which is not a multiple of $\mathbf{v}$. All the vectors

$$\lambda\mathbf{v} + \mu\mathbf{w}$$

are in S as it is closed under addition and scalar multiplication.

$\mathbf{w}$ not being a multiple of $\mathbf{v}$ is equivalent to the vectors $\mathbf{v}$ and $\mathbf{w}$ being linearly independent. One can show that for every real x, y there exist λ, μ such that

$$\lambda\mathbf{v} + \mu\mathbf{w} = (x, y)$$

– details of this appear around (3.29). It follows that $S = \mathbb{R}^2$.

In a similar fashion the subsets of $\mathbb{R}^3$ which are closed under addition and scalar multiplication and are

$$\{\mathbf{0}\}, \qquad \text{lines through the origin,} \qquad \text{planes containing the origin,} \qquad \mathbb{R}^3.$$