Solution (#516) Let $P = (1, -2, 3)$ and $Q = (-3, 0, 2)$. Points on the line PQ are of the form

$$R_\lambda = (1, -2, 3) + \lambda((-3, 0, 2) - (1, -2, 3)) = (1 - \lambda)(1, -2, 3) + \lambda(-3, 0, 2).$$

Note that $R_0 = P$ and $R_1 = Q$. Also note

$$|R_\lambda P| = |\lambda(-4, 2, -1)| = |\lambda| \sqrt{4^2 + 2^2 + 1^2} = |\lambda| \sqrt{21},$$

and

$$|R_\lambda Q| = |(\lambda - 1)(4, 2, -1)| = |\lambda - 1| \sqrt{21}.$$

The desired point R between P and Q has $0 < \lambda < 1$ and satisfies

$$\frac{2}{3} = \frac{|R_\lambda P|}{|R_\lambda Q|} = \frac{|\lambda|}{|\lambda - 1|} = \frac{\lambda}{1 - \lambda},$$

and hence $\lambda = 2/5$. We then have

$$R = R_{2/5} = \frac{3}{5}(1, -2, 3) + \frac{2}{5}(-3, 0, 2) = \frac{1}{5}(-3, -6, 13).$$

There is a second point S on the line PQ which satisfies $|PS| = \frac{2}{3}|QS|$. For this we need to find the second solution of

$$\frac{2}{3} = \frac{|\lambda|}{|\lambda - 1|}.$$

If $\lambda > 1$ the equation simplifies to

$$\frac{2}{3} = \frac{\lambda}{\lambda - 1} \implies 2\lambda - 2 = 3\lambda \implies \lambda = -2,$$

which is not in the range $\lambda > 1$. If $\lambda < 0$ the equation simplifies to

$$\frac{2}{3} = \frac{-\lambda}{1 - \lambda} \implies 2 - 2\lambda = -3\lambda \implies \lambda = -2.$$

Hence we have

$$S = R_{-2} = 3(1, -2, 3) - 2(-3, 0, 2) = (9, -6, 5).$$