Solution (#520) Fix λ. The distance from $\mathbf{r}(\lambda)$ to the line $\mathbf{s}(\mu)$ is minimal when

$$(\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(0,2,1)=0.$$

Similarly, for a fixed μ, the distance from $\mathbf{s}(\mu)$ to the line $\mathbf{r}(\lambda)$ is minimal when

$$(\mathbf{r}(\lambda)-\mathbf{s}(\mu))\cdot(2,3,2)=0.$$

For the shortest distance between the lines, these two equations are simultaneously met. So we have

$$\begin{aligned} 0 &= (-1+2\lambda, 2+3\lambda-2\mu, 2\lambda-\mu)\cdot(0,2,1) = 4+8\lambda-5\mu; \\ 0 &= (-1+2\lambda, 2+3\lambda-2\mu, 2\lambda-\mu)\cdot(2,3,2) = 4+17\lambda-8\mu. \end{aligned}$$

Solving

$$8\lambda-5\mu=-4; \qquad 17\lambda-8\mu=-4,$$

gives

$$\lambda=\frac{4}{7}; \qquad \mu=\frac{12}{7}.$$

We then have

$$\mathbf{r}\left(\frac{4}{7}\right)=\frac{1}{7}(15,33,8); \qquad \mathbf{s}\left(\frac{12}{7}\right)=\frac{1}{7}(14,31,12),$$

and

$$\mathbf{r}\left(\frac{4}{7}\right)-\mathbf{s}\left(\frac{12}{7}\right)=\frac{1}{7}(1,2,-4)$$

which has length

$$\frac{1}{7}\sqrt{1^2+2^2+(-4)^2}=\frac{1}{7}\sqrt{21}=\sqrt{\frac{3}{7}}.$$