Solution (#525) Show that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{u} &= 1, \qquad \mathbf{v}\cdot\mathbf{v}=1, \qquad \mathbf{w}\cdot\mathbf{w}=1, \\ \mathbf{u}\cdot\mathbf{v} &= 0, \qquad \mathbf{v}\cdot\mathbf{w}=0, \qquad \mathbf{u}\cdot\mathbf{w}=0. \end{aligned}$$

If $(x,y,z)=\lambda\mathbf{u}+\mu\mathbf{v}$ note

$$x+y=\left(\frac{\lambda}{\sqrt{2}}+\frac{\mu}{\sqrt{3}}\right)+\left(\frac{-\lambda}{\sqrt{2}}+\frac{\mu}{\sqrt{3}}\right)=\frac{2\mu}{\sqrt{3}}=2z$$

and conversely if $x+y=2z$ then

$$x,y,z=\left(\frac{x}{\sqrt{2}}-\frac{y}{\sqrt{2}}\right)\mathbf{u}+\left(\frac{x\sqrt{3}}{2}+\frac{y\sqrt{3}}{2}\right)\mathbf{v}$$

Finally, if $\mathbf{r}=\lambda\mathbf{u}+\mu\mathbf{v}$ then $\mathbf{r}\cdot\mathbf{u}=\lambda$ and $\mathbf{r}\cdot\mathbf{v}=\mu$ follows from $\mathbf{u}\cdot\mathbf{u}=1$, $\mathbf{u}\cdot\mathbf{v}=0$, $\mathbf{v}\cdot\mathbf{v}=1$.