Solution (#539) Say that $\mathbf{r}_1(\lambda_1,\mu_1)$ and $\mathbf{r}_2(\lambda_2,\mu_2)$ are parametrizations of the same plane Π. Say that

$$\mathbf{r}_1(\lambda_1,\mu_1) = \mathbf{p}_1 + \lambda_1\mathbf{a}_1 + \mu_1\mathbf{b}_1, \qquad \mathbf{r}_2(\lambda_2,\mu_2) = \mathbf{p}_2 + \lambda_2\mathbf{a}_2 + \mu_2\mathbf{b}_2,$$

where $\mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{a}_2, \mathbf{b}_2$ are independent pairs of vectors. As $\mathbf{r}_1$ and $\mathbf{r}_2$ parametrize the same plane there are values e and f such that

$$\mathbf{r}_1(e,f) = \mathbf{r}_2(0,0) = \mathbf{p}_2.$$

As the vectors $\mathbf{a}_2$ and $\mathbf{b}_2$ are parallel to Π then

$$\mathbf{a}_2 = \mathbf{r}_1(a,c) - \mathbf{p}_1 = a\mathbf{a}_1 + c\mathbf{b}_1, \qquad \mathbf{b}_2 = \mathbf{r}_1(b,d) - \mathbf{p}_1 = b\mathbf{a}_1 + d\mathbf{b}_1,$$

for some a, b, c, d. Hence

$$\begin{aligned}
\mathbf{r}_2(\lambda_2,\mu_2) &= \mathbf{p}_2 + \lambda_2\mathbf{a}_2 + \mu_2\mathbf{b}_2 \\
&= (\mathbf{p}_1 + e\mathbf{a}_1 + f\mathbf{b}_1) + \lambda_2(a\mathbf{a}_1 + c\mathbf{b}_1) + \mu_2(b\mathbf{a}_1 + d\mathbf{b}_1) \\
&= \mathbf{p}_1 + (e + \lambda_2 a + \mu_2 b)\mathbf{a}_1 + (f + \lambda_2 c + \mu_2 d)\mathbf{b}_1 \\
&= \mathbf{r}_1(\lambda_2 a + \mu_2 b + e, \lambda_2 c + \mu_2 d + f).
\end{aligned}$$

So if $\mathbf{r}_1(\lambda_1,\mu_1) = \mathbf{r}_2(\lambda_2,\mu_2)$, by the uniqueness of parameters, we have

$$\lambda_1 = \lambda_2 a + \mu_2 b + e, \qquad \mu_1 = \lambda_2 c + \mu_2 d + f.$$

these are two simultaneous equations giving λ_2 and μ_2 in terms of λ_1 and μ_1. For there to be a unique solution we need that $ad \neq bc$ (see (3.25) for further details if necessary.)