**Solution** (#540) Let $\mathbf{n}$ be a unit vector and let $P$ denote reflection orthogonal projection on to the plane $\mathbf{r}\cdot\mathbf{n}=0$.

The orthogonal projection $P(\mathbf{v})$ of $\mathbf{v}$ is the unique point in the plane such that the plane's normal at $P(\mathbf{v})$ passes through $\mathbf{v}$. So $P(\mathbf{v})$ is uniquely specified by

$$P(\mathbf{v})\cdot\mathbf{n}=0 \qquad \text{and} \qquad P(\mathbf{v})=\mathbf{v}+\lambda\mathbf{n} \text{ for some real number } \lambda.$$

So we have

$$(\mathbf{v}+\lambda\mathbf{n})\cdot\mathbf{n}=0 \quad \text{giving} \quad \lambda=-\mathbf{v}\cdot\mathbf{n}$$

and hence $P(\mathbf{v})=\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. Further, as $\mathbf{n}\cdot\mathbf{n}=1$,

$$\begin{aligned} P^2(\mathbf{v}) &= P(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}) \\ &= (\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n})-[(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n})\cdot\mathbf{n}]\mathbf{n} \\ &= \mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+(\mathbf{v}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{n})\mathbf{n} \\ &= \mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+(\mathbf{v}\cdot\mathbf{n})\mathbf{n} \\ &= \mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}=P(\mathbf{v}). \end{aligned}$$

Now let $R$ denote reflection in the plane $\mathbf{r}\cdot\mathbf{n}=0$. With $P$ as above we note that $P(\mathbf{v})$ is the midpoint of $Q(\mathbf{v})$ and $\mathbf{v}$. Hence

$$P(\mathbf{v})=\frac{Q(\mathbf{v})+\mathbf{v}}{2}$$

which rearranges to

$$\begin{aligned} Q(\mathbf{v}) &= 2P(\mathbf{v})-\mathbf{v} \\ &= 2(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n})-\mathbf{v} \\ &= \mathbf{v}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}. \end{aligned}$$

Note that

$$\begin{aligned} Q^2(\mathbf{v}) &= Q(\mathbf{v}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}) \\ &= \mathbf{v}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-2[(\mathbf{v}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n})\cdot\mathbf{n}]\mathbf{n} \\ &= \mathbf{v}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}-2(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+4(\mathbf{v}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{n})\mathbf{n} \\ &= \mathbf{v}-4(\mathbf{v}\cdot\mathbf{n})\mathbf{n}+4(\mathbf{v}\cdot\mathbf{n})\mathbf{n} \\ &= \mathbf{v}. \end{aligned}$$