Solution (#561) Define the complex 2×2 matrices $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ as

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad \mathcal{J} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \qquad \mathcal{K} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathcal{I}^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2; \\
\mathcal{J}^2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = -I_2; \\
\mathcal{K}^2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = -I_2.
\end{aligned}$$

Also

$$\begin{aligned}
\mathcal{I}\mathcal{J} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \mathcal{K}; \qquad \mathcal{J}\mathcal{I} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\mathcal{K}; \\
\mathcal{J}\mathcal{K} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} = \mathcal{I}; \qquad \mathcal{K}\mathcal{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i^2 \\ -i^2 & 0 \end{pmatrix} = -\mathcal{I}; \\
\mathcal{K}\mathcal{I} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \mathcal{J}; \qquad \mathcal{I}\mathcal{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\mathcal{J}.
\end{aligned}$$