Solution (#582) Say $i \neq j$. Then

$$\begin{aligned}[A+B]_{ij} &= [A]_{ij}+[B]_{ij}=0 \qquad \text{as } A \text{ and } B \text{ are diagonal.}\\ [cA]_{ij} &= c[A]_{ij} \qquad \text{as } A \text{ is diagonal.}\end{aligned}$$

As A and B are diagonal then

$$[A]_{ij}=[A]_{ii}\delta_{ij}, \qquad [B]_{ij}=[B]_{ii}\delta_{ij}.$$

Then

$$\begin{aligned}[AB]_{ij} &= \sum_{k=1}^{n}[A]_{ik}[B]_{kj}=\sum_{k=1}^{n}[A]_{ii}\delta_{ik}[B]_{jj}\delta_{kj}=[A]_{ii}[B]_{jj}\delta_{ij}.\\ [BA]_{ij} &= \sum_{k=1}^{n}[B]_{ik}[A]_{kj}=\sum_{k=1}^{n}[B]_{ii}\delta_{ik}[A]_{jj}\delta_{kj}=[B]_{ii}[A]_{jj}\delta_{ij}.\end{aligned}$$

These expressions are zero unless $i=j$ when they are equal. So AB is diagonal and $AB=BA$.

In amongst this formal notation something simple may have been lost, so it's worth remarking that the following has been shown: if

$$A=\mathrm{diag}(a_1,a_2,\ldots,a_n), \qquad B=\mathrm{diag}(b_1,b_2,\ldots,b_n),$$

then

$$\begin{aligned}A+B &= \mathrm{diag}(a_1+b_1,a_2+b_2,\ldots,a_n+b_n),\\ cA &= \mathrm{diag}(ca_1,ca_2,\ldots,ca_n),\end{aligned}$$

and

$$AB=\mathrm{diag}(a_1b_1,a_2b_2,\ldots,a_nb_n)=\mathrm{diag}(b_1a_1,b_2a_2,\ldots,b_na_n)=BA.$$