Solution (#588) (i) Let A be an $n \times n$ matrix. For $n \times 1$ column vectors $\mathbf{v}$, $\mathbf{w}$ we have $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$. So

$$A\mathbf{v} \cdot A\mathbf{w} = \mathbf{v} \cdot \mathbf{w} \quad \iff \quad (A\mathbf{v})^T (A\mathbf{w}) = \mathbf{v}^T \mathbf{w} \iff \mathbf{v}^T A^T A\mathbf{w} = \mathbf{v}^T \mathbf{w}.$$

If $A^T A = I_n$ it follows that $A\mathbf{v} \cdot A\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$.

Conversely, suppose that $A\mathbf{v} \cdot A\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $n \times 1$ column vectors. In particular when $\mathbf{v} = \mathbf{e}_i^T$, and $\mathbf{w} = \mathbf{e}_j^T$ then #563 (iii) shows that

$$[A^T A]_{ij} = \mathbf{e}_i A^T A \mathbf{e}_j^T = (A\mathbf{e}_i^T)^T (A\mathbf{e}_j^T) = (A\mathbf{e}_i^T) \cdot (A\mathbf{e}_j^T) = \mathbf{e}_i^T \cdot \mathbf{e}_j^T = \delta_{ij}$$

and hence $A^T A = I_n$.

(ii) Let A be an $n \times n$ matrix. Suppose that A is orthogonal. The jth column of A is $A\mathbf{e}_j^T$ and we have

$$(A\mathbf{e}_i^T) \cdot (A\mathbf{e}_j^T) = (A\mathbf{e}_i^T)^T (A\mathbf{e}_j^T) = \mathbf{e}_i A^T A \mathbf{e}_j^T = \mathbf{e}_i^T \cdot \mathbf{e}_j^T = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

It follows from this that the columns are of unit length and mutually perpendicular.

Conversely suppose that the columns of A are mutually perpendicular unit vectors. Then, by #563 (iii)

$$\delta_{ij} = (A\mathbf{e}_i^T) \cdot (A\mathbf{e}_j^T) = (A\mathbf{e}_i^T)^T (A\mathbf{e}_j^T) = \mathbf{e}_i A^T A \mathbf{e}_j^T = \mathbf{e}_i A^T A \mathbf{e}_j^T = [A^T A]_{ij}$$

and hence $A^T A = I_n$.

(iii) Let A be an $n \times n$ matrix such that $|A\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x}$. Recall that $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. So for any vectors $\mathbf{v}$ and $\mathbf{w}$ we have

$$A(\mathbf{v} + \mathbf{w}) \cdot A(\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}).$$

This expands to

$$(A\mathbf{v}) \cdot (A\mathbf{v}) + 2(A\mathbf{v}) \cdot (A\mathbf{w}) + (A\mathbf{w}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}.$$

As $|A\mathbf{v}|^2 = (A\mathbf{v}) \cdot (A\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ and similarly $(A\mathbf{w}) \cdot (A\mathbf{w}) = \mathbf{w} \cdot \mathbf{w}$ then

$$(A\mathbf{v}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$. It follows that A is orthogonal by (i).

Conversely if A is orthogonal then

$$|A\mathbf{x}|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T A^T A \mathbf{x} = \mathbf{x}^T \mathbf{x} = |\mathbf{x}|^2.$$