Solution (#610) The different ways in which three planes might fail to intersect in a unique point were listed as:

(a) the three planes are all same;
(b) three planes meet in a line;
(c) two planes are parallel and distinct with the third plane meeting both;
(d) the three planes pairwise meet in distinct parallel lines;
(e) three planes all parallel and not all the same.

Note firstly that two planes $ax+by+cz=d$ and $Ax+By+Cz=D$ are parallel if and only if their normal vectors (a,b,c) and (A,B,C) are multiples of one another. They are the same plane if the two equations are multiples of one another.

(i) $x-2y=z+1;\quad 2x+y=3;\quad x=2y+z$. As the first and third planes are distinct but both have normals parallel to $(1,-2,-1)$, and the second plane is parallel to neither, then this is case (c).

(ii) $2x+y=3z+2;\quad x+2z=3y+1;\quad y+z+2=2x$. No pair of planes is parallel. If we look to solve these equations, say by using the first equation as $y=3z-2x+2$ to eliminate y from the second and third equations we get

$$7x-7z=7;\quad 4z+4=4x.$$

Together these equations are equivalent to $z=x-1$ and then $y=x-1$ also. So our three planes meet in the line

$$x-1=y=z.$$

This is case (b).

(iii) $2x+3y=3z-2;\quad x+y+1=z;\quad x+2y=2z$. No pair of planes is parallel. If we look to solve these equations though, say by using the second equation to eliminate z from the first and third equations, we obtain

$$x=-1;\quad x=-2.$$

The three planes have no common intersection and this is case (d). Further investigation shows that the three parallel lines in which they meet are:

$$(-1,0,0)+\lambda(0,1,1);\qquad (-2,1,0)+\lambda(0,1,1);\qquad (-4,2,0)+\lambda(0,1,1).$$

(iv) $x+y+z=4;\quad x+2y=3;\quad x+2z=1$. No pair of planes is parallel. If we look to solve these equations though, say by using the first equation as $x=4-y-z$ to eliminate x from the second and third equations, we obtain

$$-z+y=-1;\quad -y+z=-3,$$

which are clearly contradictory and the three planes have no common intersection. This is again case (d). Further investigation shows that the three parallel lines in which they meet are:

$$(5,-1,0)+\lambda(2,-1,-1);\qquad (1,3,0)+\lambda(2,-1,-1);\qquad (1,1,0)+\lambda(2,-1,-1).$$