Solution (#638) Let

$$A=\begin{pmatrix}1&1\\3&4\\2&3\end{pmatrix};\qquad \mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix};\qquad \mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}.$$

(i) We have

$$\begin{pmatrix}1&1\\3&4\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\3x+4y\\2x+3y\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix};$$

if we substitute $x=-y$ into the second equation we get $y=0$ and so $x=0$ also.
(ii) Now

$$\begin{aligned}B_1A &= \begin{pmatrix}8&-5&4\\-6&4&-3\end{pmatrix}\begin{pmatrix}1&1\\3&4\\2&3\end{pmatrix}=\begin{pmatrix}8-15+8&8-20+12\\-6+12-6&-6+16-9\end{pmatrix}=I_2;\\ B_2A &= \begin{pmatrix}-4&7&-8\\3&-5&6\end{pmatrix}\begin{pmatrix}1&1\\3&4\\2&3\end{pmatrix}=\begin{pmatrix}-4+21-16&-4+28-24\\3-15+12&3-20+18\end{pmatrix}=I_2.\end{aligned}$$

(iii) Now consider the system

$$\begin{pmatrix}1&1\\3&4\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\3x+4y\\2x+3y\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}.$$

From the first two equations we can deduce that

$$x=4b_1-b_2,\qquad y=b_2-3b_1.$$

So there is clearly at most one solution and these values of x,y will satisfy the third equation if and only if

$$2(4b_1-b_2)+3(b_2-3b_1)=b_3 \iff b_1+b_3=b_2.$$

(iv) If $\mathbf{b}=(2,7,5)^T$ then this condition is met as $2+5=7$. We see from (iii) that our solution is $(x,y)=(1,1)$. Note also that

$$\begin{aligned}B_1\mathbf{b} &= \begin{pmatrix}8&-5&4\\-6&4&-3\end{pmatrix}\begin{pmatrix}2\\7\\5\end{pmatrix}=\begin{pmatrix}16-35+20\\-12+28-15\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix};\\ B_2\mathbf{b} &= \begin{pmatrix}-4&7&-8\\3&-5&6\end{pmatrix}\begin{pmatrix}2\\7\\5\end{pmatrix}=\begin{pmatrix}-8+49-40\\6-35+30\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\end{aligned}$$

though this should not be surprising as $A\mathbf{x}=\mathbf{b}$ implies $\mathbf{x}=I_2\mathbf{x}=BA\mathbf{x}=B\mathbf{b}$ for any left inverse of A.
(v) As x and y vary, the point $A\mathbf{x}$ ranges over all those $\mathbf{b}$ that are consistent with $A\mathbf{x}=\mathbf{b}$, i.e. over the plane

$$b_1+b_3=b_2.$$