Solution (#642) (i) Say $AX = B$ where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

As A is invertible with

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix},$$

then

$$X = A^{-1}B = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -10 & -6 \\ 8 & 5 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ 4 & 2\frac{1}{2} \end{pmatrix}.$$

(ii) Say now that $AX = B$ where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}.$$

This time A is not invertible. If we write

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the single matrix equation $AX = B$ is equivalent to the scalar equations

$$a + c = 3; \qquad b + d = 2; \qquad 2a + 2c = 6; \qquad 2b + 2d = 6.$$

Clearly the third and fourth equations follow from the first and second and so the general solution for X is

$$X = \begin{pmatrix} a & b \\ 3 - a & 2 - b \end{pmatrix}.$$

(iii) Say $AX = B$ where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Again A is not invertible. If we write the entries for X as before we obtain the four equations

$$a + 2c = 1; \qquad b + 2d = 0; \qquad 2a + 4c = 0; \qquad 2b + 4d = 1.$$

Note that the third equation contradicts the first (as the fourth does the second) so there are no matrices X solving $AX = B$.