Solution (#656) (i) Let A be an $m\times n$ matrix where $1\leqslant m<n$ and consider the homogeneous linear system $(A\,|\,\mathbf{0})$.

If we reduce the system we eventually arrive at $(\mathrm{RRE}(A)\,|\,\mathbf{0})$, and as $m<n$ then $\mathrm{RRE}(A)$ has at most m leading 1s and hence fewer than n leading 1s. The solutions for the system can be described by assigning parameters to each column/variable without a leading 1 and there is at least one such column. Hence there are infinitely many solutions.

(ii) Let A be an $m\times n$ matrix such that the linear system $A\mathbf{x}=\mathbf{b}$ is consistent for all column vectors $\mathbf{b}$ in $\mathbb{R}^n$. If we reduce the system we arrive at

$$\mathrm{RRE}(A)\,\mathbf{x}=E\mathbf{b} \tag{10.17}$$

for some invertible matrix E such that $\mathrm{RRE}(A)=EA$, and the system (10.17) has precisely the same solutions as the original system. If $m>n$ then $\mathrm{RRE}(A)$ necessarily has a zero row as its bottom row and this means that the system (10.17) is inconsistent when $E\mathbf{b}=\mathbf{e}_n^T$ or equivalently when $\mathbf{b}=E^{-1}\mathbf{e}_n^T$ contradicting our hypotheses. Hence $m\leqslant n$.