Solution (#658) We may write this linear system as $(A\,|\,B)$ and apply EROs. So we find

$$\left(\begin{array}{ccc|ccc} 1 & 1 & a & 1 & 0 & b \\ 2 & 1 & -1 & -1 & b & -1 \\ 1 & 1 & 0 & b & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & b & 0 & 1 \\ 2 & 1 & -1 & -1 & b & -1 \\ 1 & 1 & a & 1 & 0 & b \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & b & 0 & 1 \\ 0 & 1 & 1 & 1+2b & -b & 3 \\ 0 & 0 & a & 1-b & 0 & b-1 \end{array}\right).$$

If $a \neq 0$ then we can see that we will have a unique solution. In this case we can continue reducing as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & b & 0 & 1 \\ 0 & 1 & 1 & 1+2b & -b & 3 \\ 0 & 0 & 1 & (1-b)/a & 0 & (b-1)/a \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & (1-a-ab-b)/a & b & (b-2a-1)/a \\ 0 & 1 & 0 & (a+2ab-1+b)/a & -b & (3a-b+1)/a \\ 0 & 0 & 1 & (1-b)/a & 0 & (b-1)/a \end{array}\right).$$

So when $a \neq 0$ then we get a unique solution of

$$X = \frac{1}{a}\begin{pmatrix} 1-a-ab-b & ab & b-2a-1 \\ a+2ab+b-1 & -ab & 3a-b+1 \\ 1-b & 0 & b-1 \end{pmatrix}.$$

Now if $a=0$ then we see we only have consistency if $b=1$. For this case we need to solve the system

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & -2 \\ 0 & 1 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and we see that the general solution for X is

$$X = \begin{pmatrix} -2+\alpha & 1+\beta & -2+\gamma \\ 3-\alpha & -1-\beta & 3-\gamma \\ \alpha & \beta & \gamma \end{pmatrix}$$

where α, β, γ are arbitrary constants.