Solution (#661) (i) Let

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \qquad B_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

If we consider the equation $A_1X + XB_1 = C_1$,

$$\begin{pmatrix} 3x_{11}+x_{21} & 2x_{11}+3x_{12}+x_{22} \\ x_{11}+3x_{21} & x_{12}+2x_{21}+3x_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

as a linear system in the entries x_{ij} of X we have

$$\left(\begin{array}{cccc|c} 3 & 0 & 1 & 0 & c_{11} \\ 2 & 3 & 0 & 1 & c_{12} \\ 1 & 0 & 3 & 0 & c_{21} \\ 0 & 1 & 2 & 3 & c_{22} \end{array}\right).$$

The 4×4 matrix reduces as

$$\begin{pmatrix} 3 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{pmatrix} \longrightarrow I_4.$$

Hence the system has a unique solution.

(ii) For the following A_2 and B_2,

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

the equation $A_2X + XB_2 = C_2$

$$\begin{pmatrix} 3x_{11}+x_{21} & 2x_{11}+x_{12}+x_{22} \\ x_{11}+3x_{21} & x_{12}+2x_{21}+x_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

give a system

$$\left(\begin{array}{cccc|c} 3 & 0 & 1 & 0 & c_{11} \\ 2 & 1 & 0 & 1 & c_{12} \\ 1 & 0 & 3 & 0 & c_{21} \\ 0 & 1 & 2 & 1 & c_{22} \end{array}\right).$$

This system reduces as

$$\left(\begin{array}{cccc|c} 3 & 0 & 1 & 0 & c_{11} \\ 2 & 1 & 0 & 1 & c_{12} \\ 1 & 0 & 3 & 0 & c_{21} \\ 0 & 1 & 2 & 1 & c_{22} \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & c_{21} \\ 0 & 1 & -6 & 1 & c_{12}-2c_{21} \\ 0 & 0 & -8 & 0 & c_{11}-3c_{21} \\ 0 & 1 & 2 & 1 & c_{22} \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & c_{21} \\ 0 & 1 & -6 & 1 & c_{12}-2c_{21} \\ 0 & 0 & -8 & 0 & c_{11}-3c_{21} \\ 0 & 0 & 8 & 0 & c_{22}-c_{12}+2c_{21} \end{array}\right).$$

Hence we have a consistent system only when

$$c_{11} - c_{12} - c_{21} + c_{22} = 0.$$

This does not hold when $C = I_2$ and hence $AX + XB = I_2$ has no solution X for this C.