Solution (#666) Reducing $(A\,|\,I)$ we obtain

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 1 & -2 & 4 & 0 & 1 & 0 \\ 2 & -5 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{A_{12}(-1)} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 2 & -5 & 5 & 0 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{A_{13}(-2)} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 1 & 5 & -2 & 0 & 1 \end{array}\right) \xrightarrow{A_{23}(-1)} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$
$$\xrightarrow{A_{32}(-4)} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{A_{21}(3)} . \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 15 & -12 \\ 0 & 1 & 0 & 3 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right).$$

Hence the inverse is

$$A^{-1} = \begin{pmatrix} 10 & 15 & -12 \\ 3 & 5 & -4 \\ -1 & -1 & 1 \end{pmatrix}.$$

As

$$A_{21}(3) \cdot A_{32}(-4) \cdot A_{23}(-1) \cdot A_{13}(-2) \cdot A_{12}(-1) \cdot A = I_3$$

then

$$\begin{aligned} A^{-1} &= A_{12}(-1)^{-1} \cdot A_{13}(-2)^{-1} \cdot A_{23}(-1)^{-1} \cdot A_{32}(-4)^{-1} \cdot A_{21}(3)^{-1} \\ &= A_{12}(1) \cdot A_{13}(2) \cdot A_{23}(1) \cdot A_{32}(4) \cdot A_{21}(-3)\,. \end{aligned}$$