Solution (#674) Let A_n denote the $n \times n$ matrix

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}}_{n\times n}.$$

We note that $A_1 = I_1$ is invertible, that A_2 clearly isn't (first and second rows equal) and that A_3 reduces as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that A_3 is invertible.

More generally consider A_n where $n \geqslant 4$. We will take care with using EROs and ECOs (elementary column operations, achieved by postmultiplication with elementary matrices) to show that A_n is invertible if and only if A_{n-3} is. (As the effect of an ECO is to postmultiply by an elementary matrix an ECO will not affect a matrix's invertibility.) Focusing on the bottom three rows and rightmost three columns we see

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 1 & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 1 & 1 \\ \cdots & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{ERO}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 1 & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{ECO}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 1 & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{ERO}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ECO}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ECO}} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \ddots & 1 & 1 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So using EROs and ECOs we have transformed A_n into $\operatorname{diag}(A_{n-3}, I_3)$.

This means that the invertibility (or otherwise) of A_n repeats with a period of three. As A_1 and A_3 are invertible and A_2 is singular, this means that A_n is singular when $n+1$ is a multiple of three.