Solution (#679) Let $A = (a_{ij})$ be an $m \times n$ matrix where $m \leqslant n$. We are assuming that the swapping of rows and columns takes negligible computational time and so we may assume that the matrix has rank m with the leading 1s appearing in the first m columns as we row-reduce the matrix. In practice, should a matrix include zero rows, then this would only lessen the time needed to reduce it – we are here considering a worst-case scenario.

In the first stage, when clearing out the first column, we perform n divisions to make the leading entry of the first row equal 1 and replace each of a_{1j} with a_{1j}/a_{11}. We then perform $(m-1)n$ subtractions and multiplications to subtract a_{i1} times the first row from the ith row; that is we make the replacements

$$a_{ij} \mapsto a_{ij} - a_{i1}\left(\frac{a_{1j}}{a_{11}}\right) \qquad 2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

At the second stage, when clearing out the second column, we perform $n-1$ divisions on the second row to make its leading entry equal 1 (the first entry is already 0). We then perform $(m-1)(n-1)$ subtractions and $(m-1)(n-1)$ multiplications to subtract a_{i2} times the second from each of the other $m-1$ rows.

Hence at the ith stage (where $1 \leqslant i \leqslant m$) we perform

$$n-i+1 \text{ divisions}, \qquad (m-1)(n-i+1) \text{ subtractions}, \qquad (m-1)(n-i+1) \text{ multiplications}.$$

Thus, overall, we make

$$\begin{aligned}
\sum_{i=1}^{m}(n-i+1) &= \frac{1}{2}m(2n-m+1) && \text{divisions;}\\
\sum_{i=1}^{m}(m-1)(n-i+1) &= \frac{1}{2}m(m-1)(2n-m+1) && \text{subtractions;}\\
\sum_{i=1}^{m}(m-1)(n-i+1) &= \frac{1}{2}m(m-1)(2n-m+1) && \text{multiplications.}
\end{aligned}$$

The total number of arithmetic operations equals

$$\frac{1}{2}m(2m-1)(2n-m+1),$$

which we can take as our choice of $P(m,n)$. Note that

$$P(n,n) = n^3 + \frac{n^2}{2} - \frac{n}{2}$$

so that $P(n,n)/n^3$ converges to 1 as n becomes large.

Finally an upper bound for the number of arithmetic operations needed to invert an $n \times n$ matrix using Algorithm 3.84 equals

$$P(n,2n) = \frac{1}{2}n(2n-1)(4n-n+1) = \frac{1}{2}n(2n-1)(3n+1).$$