Solution (#683) Let i, j, k be distinct and let $\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k$ denote the rows of a matrix to which we will apply the EROs $A_{ij}(\lambda)$ and $A_{ik}(\mu)$. Performing first $A_{ij}(\lambda)$ then $A_{ik}(\mu)$ we have

$$\begin{array}{ccccc} \mathbf{r}_i & \longrightarrow & \mathbf{r}_i & \longrightarrow & \mathbf{r}_i; \\ \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j; \\ \mathbf{r}_k & \longrightarrow & \mathbf{r}_k & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_k. \end{array}$$

Whereas performing first $A_{ik}(\lambda)$ then $A_{ij}(\mu)$ we have

$$\begin{array}{ccccc} \mathbf{r}_i & \longrightarrow & \mathbf{r}_i & \longrightarrow & \mathbf{r}_i; \\ \mathbf{r}_j & \longrightarrow & \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j; \\ \mathbf{r}_k & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_k & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_k. \end{array}$$

So we see the EROs commute, all other rows being unaffected.

However more generally $A_{ij}(\lambda)$ and $A_{jk}(\mu)$ don't commute. Performing first $A_{ij}(\lambda)$ and then $A_{jk}(\mu)$ we have

$$\begin{array}{ccccc} \mathbf{r}_i & \longrightarrow & \mathbf{r}_i & \longrightarrow & \mathbf{r}_i; \\ \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j; \\ \mathbf{r}_k & \longrightarrow & \mathbf{r}_k & \longrightarrow & \lambda\mu\mathbf{r}_i + \mu\mathbf{r}_j + \mathbf{r}_k, \end{array}$$

whilst performing first $A_{jk}(\mu)$ and then $A_{ij}(\lambda)$ we have

$$\begin{array}{ccccc} \mathbf{r}_i & \longrightarrow & \mathbf{r}_i & \longrightarrow & \mathbf{r}_i; \\ \mathbf{r}_j & \longrightarrow & \mathbf{r}_j & \longrightarrow & \lambda\mathbf{r}_i + \mathbf{r}_j; \\ \mathbf{r}_k & \longrightarrow & \mu\mathbf{r}_j + \mathbf{r}_k & \longrightarrow & \mu\mathbf{r}_j + \mathbf{r}_k. \end{array}$$

In general we can see that these effects are different from one another and that

$$A_{ij}(\lambda)A_{jk}(\mu) \neq A_{jk}(\mu)A_{ij}(\lambda).$$