Solution (#692) Let

$$A = \begin{pmatrix} 1 & 4 & -2 & 1 \\ -1 & -6 & 8 & -5 \\ 2 & 4 & 8 & -6 \end{pmatrix}.$$

We can row-reduce this, and keep track of our EROs, by reducing $(A\,|\,I_3)$ as follows.

$$\left(\begin{array}{cccc|ccc} 1 & 4 & -2 & 1 & 1 & 0 & 0 \\ -1 & -6 & 8 & -5 & 0 & 1 & 0 \\ 2 & 4 & 8 & -6 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 4 & -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -4 & 1 & 1 & 0 \\ 0 & -4 & 12 & -8 & -2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|ccc} 1 & 4 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -4 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 0 & 10 & -7 & 3 & 2 & 0 \\ 0 & 1 & -3 & 2 & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -4 & -2 & 1 \end{array}\right).$$

So we have

$$PA = \begin{pmatrix} 1 & 0 & 10 & -7 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{where} \qquad P = \begin{pmatrix} 3 & 2 & 0 \\ -1/2 & -1/2 & 0 \\ -4 & -2 & 1 \end{pmatrix}.$$

Now we can column reduce PA as follows

$$\left(\begin{array}{cccc} 1 & 0 & 10 & -7 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -10 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Finally then

$$PAQ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{where} \qquad Q = \begin{pmatrix} 1 & 0 & -10 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$