Solution (#711) Let A be an $m \times n$ matrix and E be an $m \times m$ elementary matrix. We shall show that a linear independency involving k rows of A corresponds to a linear independency of k rows of EA. Denote the rows of A as $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and the rows of EA as $\mathbf{s}_1, \ldots, \mathbf{s}_m$.Say without loss of generality that the first k rows of A are linearly independent.

(i) Say that $E = S_{IJ}$ with $I < J$. If $k < I$ or $J \leqslant k$ then the first k rows of EA are the same as the first k rows of A (albeit possibly in a different order) and so are still linearly independent. If $I \leqslant k < J$ then the rows

$$\mathbf{s}_1, \ldots, \mathbf{s}_{I-1}, \mathbf{s}_{I+1}, \ldots, \mathbf{s}_k, \mathbf{s}_J$$

are the first k rows of A, in a different order, and so are linearly independent.

(ii) Say that $E = M_I(\lambda)$ where $\lambda \neq 0$. If $I > k$ then the first k rows of EA are the same as the first k rows of A and so are still linearly independent. If $I \leqslant k$ then

$$\begin{aligned} & \alpha_1 \mathbf{s}_1 + \cdots + \alpha_{I-1}\mathbf{s}_{I-1} + \alpha_I \mathbf{s}_I + \alpha_{I+1}\mathbf{s}_{I+1} + \cdots + \alpha_k \mathbf{s}_k = \mathbf{0} \\ \iff \quad & \alpha_1 \mathbf{r}_1 + \cdots + \alpha_{I-1}\mathbf{r}_{I-1} + \alpha_I \lambda \mathbf{r}_I + \alpha_{I+1}\mathbf{r}_{I+1} + \cdots + \alpha_k \mathbf{r}_k = \mathbf{0} \\ \iff \quad & \alpha_1 = \cdots = \alpha_{I-1} = \alpha_I \lambda = \alpha_{I+1} = \alpha_k = 0 \\ \iff \quad & \alpha_1 = \cdots = \alpha_{I-1} = \alpha_I = \alpha_{I+1} = \alpha_k = 0. \end{aligned}$$

So the first first k rows of EA are again linearly independent.

(iii) Say that $E = A_{IJ}(\lambda)$ where $\lambda \neq 0$. (Note there is nothing to prove if $\lambda = 0$ as E is the identity matrix.). If $J > k$ then the first k rows of EA are the same as the first k rows of A and so are still linearly independent. If $J \leqslant k$ and the first k rows of EA are independent we are done; otherwise there is a non-trivial dependency in the rows $\mathbf{s}_1, \ldots, \mathbf{s}_k$ in which case we claim that $\mathbf{s}_1, \ldots, \mathbf{s}_{J-1}, \mathbf{s}_{J+1}, \ldots, \mathbf{s}_k, \mathbf{s}_I$ are independent.

This linear dependency is of the form

$$c_1 \mathbf{r}_1 + \cdots + c_{J-1}\mathbf{r}_{J-1} + c_J(\mathbf{r}_J + \lambda \mathbf{r}_I) + c_{I+1}\mathbf{r}_{J+1} + \cdots + c_k \mathbf{r}_k = \mathbf{0}.$$

It must be that $c_J \neq 0$ as $\mathbf{r}_1, \ldots, \mathbf{r}_{J-1}, \mathbf{r}_{J+1}, \ldots, \mathbf{r}_k$ are independent. So without loss of generality we can assume $c_J = -1$ so that

$$\lambda \mathbf{r}_I = c_1 \mathbf{r}_1 + \cdots + c_{J-1}\mathbf{r}_{J-1} - \mathbf{r}_J + c_{I+1}\mathbf{r}_{J+1} + \cdots + c_k \mathbf{r}_k = \mathbf{0}.$$

Say now that

$$\alpha_1 \mathbf{s}_1 + \cdots + \alpha_{J-1}\mathbf{s}_{J-1} + \alpha_{J+1}\mathbf{s}_{J+1} + \cdots + \alpha_k \mathbf{s}_k + \alpha_I \mathbf{s}_I = \mathbf{0}$$

or equivalently

$$\alpha_1 \mathbf{r}_1 + \cdots + \alpha_{J-1}\mathbf{r}_{J-1} + \alpha_{J+1}\mathbf{r}_{J+1} + \cdots + \alpha_k \mathbf{r}_k + \alpha_I \mathbf{r}_I = \mathbf{0}$$

as the ERO changed none of these rows. By the dependency above we have

$$\alpha_1 \mathbf{r}_1 + \cdots + \alpha_{J-1}\mathbf{r}_{J-1} + \alpha_{J+1}\mathbf{r}_{J+1} + \cdots + \alpha_k \mathbf{r}_k + \frac{\alpha_I}{\lambda}\left(c_1 \mathbf{r}_1 + \cdots + c_{J-1}\mathbf{r}_{J-1} - \mathbf{r}_J + c_{I+1}\mathbf{r}_{J+1} + \cdots + c_k \mathbf{r}_k\right) = \mathbf{0}.$$

Rearranging this means that

$$\left(\alpha_1 + \frac{\alpha_I c_1}{\lambda}\right)\mathbf{r}_1 + \cdots + \left(\alpha_{J-1} + \frac{\alpha_I c_{J-1}}{\lambda}\right)\mathbf{r}_{J-1} - \frac{\alpha_I}{\lambda}\mathbf{r}_J + \left(\alpha_{J+1} + \frac{\alpha_I c_{J+1}}{\lambda}\right)\mathbf{r}_{J+1} + \cdots + \left(\alpha_k + \frac{\alpha_k c_k}{\lambda}\right)\mathbf{r}_k = \mathbf{0}.$$

As the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_k$ are independent then

$$\alpha_1 + \frac{\alpha_I c_1}{\lambda} = \cdots = \alpha_{J-1} + \frac{\alpha_I c_{J-1}}{\lambda} = -\frac{\alpha_I}{\lambda} = \alpha_{J+1} + \frac{\alpha_I c_{J+1}}{\lambda} = \cdots = \alpha_k + \frac{\alpha_k c_k}{\lambda} = 0$$

from which it follows that

$$\alpha_1 = \cdots = \alpha_k = 0$$

and that $\mathbf{s}_1, \ldots, \mathbf{s}_{J-1}, \mathbf{s}_{J+1}, \ldots, \mathbf{s}_k, \mathbf{s}_I$ are independent.

So in each case we have shown that if A has k independent rows then EA has k independent rows. Recall also that the inverse of an elementary matrix is an elementary matrix.

Now the first r rows of $\mathrm{RRE}(A)$ are independent and so there are r independent rows in A as a series of elementary matrices can transform $\mathrm{RRE}(A)$ to A. However, if $r < m$ then any $r+1$ rows of $\mathrm{RRE}(A)$ will be dependent as they will include a zero row. Any independency of $r+1$ rows of A would lead to such an independency in $\mathrm{RRE}(A)$ – a contradiction – and hence any $r+1$ rows of A are dependent.