Solution (#712) Let $\mathbf{v}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Let A be the matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and B be the matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, $\mathbf{v}$.

Suppose that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, say

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k.$$

Then one might begin the process of row-reducing B by subtracting α_1 of the first row from the last, then subtracting α_2 of the second row from the first and so on until the last row is zero. At this point B has reduced to the matrix A with an extra zero row. By the uniqueness of RRE form we can see that $\mathrm{RRE}(B)$ is $\mathrm{RRE}(A)$ with an extra zero row. In particular $\mathrm{rank}(A) = \mathrm{rank}(B)$.

Conversely suppose that $\mathrm{rank}(A) = \mathrm{rank}(B)$. If we employ the EROs that reduce A to $\mathrm{RRE}(A)$ to the first k rows of B then we arrive at

$$\begin{pmatrix} \mathrm{RRE}(A) \\ \hline \mathbf{v} \end{pmatrix}.$$

$\mathrm{RRE}(A)$ has $\mathrm{rank}(A)$ non-zero rows which are linearly independent. If we reduce B further we must end up with the same number of zero rows, i.e. that $\mathbf{v}$ must reduce to a zero row. This means (Corollary 193) that $\mathbf{v}$ must be a linear combination of the non-zero rows of $\mathrm{RRE}(A)$ and as they are linear combinations of the rows of A (Proposition 191) then $\mathbf{v}$ also is a linear combination of those rows.