Solution (#715) Let A and B be $m \times n$ matrices. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ denote the rows of A and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ denote the rows of B.

Suppose that $\text{Row}(A+B) = \text{Row}(A)$. Then $\mathbf{a}_1 + \mathbf{b}_1$ can be written as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_m$ and so $\mathbf{b}_1$ can be written as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_m$. That is $\mathbf{b}_1$ is in $\text{Row}(A)$. This is true for each such $\mathbf{b}_i$ and hence $\text{Row}(B)$ is contained in $\text{Row}(A)$.

But the converse is not true. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix}.$$

Then $\text{Row}(B)$ is contained in $\text{Row}(A)$ (in fact they're equal) but $\text{Row}(A+B)$ is the zero space.