Solution (#719) Let A be an $m \times n$ matrix and B an $n \times m$ matrix where $m < n$.

By #656 there exists non-zero $\mathbf{x}$ in $\mathbb{R}_n$ such that $A\mathbf{x} = \mathbf{0}$ and hence $BA\mathbf{x} = \mathbf{0}$ also. By Proposition 3.95 BA is not invertible.

It is however possible for AB to be invertible. Indeed if

$$A = \begin{pmatrix} I_m & 0_{m(n-m)} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} I_m \\ 0_{(n-m)m} \end{pmatrix}$$

then we have $AB = I_m$.

Suppose then that AB is invertible. Then there exists an invertible $m \times m$ matrix M such that

$$MAB = I_m = ABM.$$

So MA is a left inverse for the $n \times m$ matrix B and hence by #709(ii) the row rank of B is m. Likewise BM is a right inverse for the $m \times n$ matrix A and so by #709(vii) the row rank of A is m.