Solution (#722) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be k independent vectors in $\mathbb{R}^n$ and $\mathbf{v}$ be a further vector in $\mathbb{R}^n$.

Suppose that $\mathbf{v}$ is not in $\langle \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$ and that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k + \alpha \mathbf{v} = \mathbf{0},$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha$. If $\alpha \neq 0$ then

$$\mathbf{v} = -\frac{1}{\alpha} \left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k \right)$$

is in the span $\langle \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$, a contradiction. So $\alpha = 0$ and then $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ by the independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We have shown $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are independent.

Conversely say $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$ are independent. If $\mathbf{v}$ is in $\langle \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$ then there exist scalars $\beta_1, \beta_2, \ldots, \beta_k$ such that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_k \mathbf{v}_k.$$

So

$$\beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_k \mathbf{v}_k - \mathbf{v} = \mathbf{0},$$

contradicting the independence of $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}$.