Solution (#733) Only S_1 and S_5 were spanning sets in #732.

$S_1 = \{(0,1,0,0),(0,0,1,0),(0,1,1,0),(3,0,0,4),(4,0,0,3)\}$. We saw in the solution to #732 that this set is spanning but, having five elements it clearly cannot be linearly independent in $\mathbb{R}^4$. If we label these vectors as $\mathbf{v}_1,\ldots,\mathbf{v}_5$ then we note that

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3.$$

This is (essentially) the only dependency between the five vectors – any further dependency between the vectors which is not simply a scalar multiple of this one would mean that the five vectors were contained in a three dimensional subset.

Any subset of S_1 which is a basis must contain four elements. Also, because of the dependency above, it must only contain two of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So the following subsets of S_1 are bases for $\mathbb{R}^4$.

$$\{\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\}, \qquad \{\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\}, \qquad \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_4,\mathbf{v}_5\}.$$

$S_5 = \{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$. If we label these vectors as $\mathbf{w}_1,\ldots\mathbf{w}_6$ then note that

$$\mathbf{w}_1 + \mathbf{w}_6 = \mathbf{w}_3 + \mathbf{w}_4 = \mathbf{w}_2 + \mathbf{w}_5.$$

Again any dependencies in $\mathbf{w}_1,\ldots\mathbf{w}_6$ can be deduced from these. If we were to remove $\mathbf{w}_1$ and $\mathbf{w}_6$ from S_5 then the dependency

$$\mathbf{w}_3 + \mathbf{w}_4 = \mathbf{w}_2 + \mathbf{w}_5$$

would still remain and so we would not have a basis. However removal of $\mathbf{w}_i$ and $\mathbf{w}_j$ from two different sides of the above equalities means no further dependency remains. There are $\binom{6}{2} = 15$ ways of removing two vectors from S_5. As noted above removal of $\mathbf{w}_1,\mathbf{w}_6$ or of $\mathbf{w}_3,\mathbf{w}_4$ or of $\mathbf{w}_2,\mathbf{w}_5$ leaves a dependency remaining. However the other 12 subsets, namely

$$\begin{array}{llll}
\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4\}, & \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_5\}, & \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_6\}, & \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_4,\mathbf{w}_5\}, \\
\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_4,\mathbf{w}_6\}, & \{\mathbf{w}_1,\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_5\}, & \{\mathbf{w}_1,\mathbf{w}_3,\mathbf{w}_5,\mathbf{w}_6\}, & \{\mathbf{w}_1,\mathbf{w}_4,\mathbf{w}_5,\mathbf{w}_6\}, \\
\{\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_6\}, & \{\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_5,\mathbf{w}_6\}, & \{\mathbf{w}_2,\mathbf{w}_4,\mathbf{w}_5,\mathbf{w}_6\}, & \{\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_5,\mathbf{w}_6\},
\end{array}$$

are all bases for $\mathbb{R}^4$.