Solution (#735) Let A, B be $n \times n$ matrices. Show that the following are subspaces of M_{nn}.

(i) Let $U_1 = \{X \in M_{nn} : AX = 0\}$. Note that 0_{nn} is in U_1. Further if X and Y are in U_1 and α, β are real scalars then

$$A(\alpha X + \beta Y) = \alpha AX + \beta AY = \alpha 0 + \beta 0 = 0,$$

and hence $\alpha X + \beta Y$ is in U_1.

(ii) Let $U_2 = \{X \in M_{nn} : AX = XA\}$. Note that 0_{nn} is in U_2. Further if X and Y are in U_2 and α, β are real scalars then

$$\begin{aligned} A(\alpha X + \beta Y) &= \alpha AX + \beta AY \\ &\alpha XA + \beta YA \\ &= (\alpha X + \beta Y)A \end{aligned}$$

and hence $\alpha X + \beta Y$ is in U_2.

(iii) Let $U_3 = \{X \in M_{nn} : AX + XB = 0\}$. Note that 0_{nn} is in U_3. Further if X and Y are in U_3 and α, β are real scalars then

$$A(\alpha X + \beta Y) + (\alpha X + \beta Y)B = \alpha(AX + XB) + \beta\,(AY + YB) = \alpha 0 + \beta 0 = 0,$$

and hence $\alpha X + \beta Y$ is in U_3.