Solution (#736) Let A be an $n \times n$ matrix and let $V_A = \{B \in M_{nn} \colon BA = AB\}$. That is, V_A is the set of $n \times n$ matrices which commute with A. By #735 we know V_A is a subspace.
(i) Let X, Y be in V_A. Then

$$\begin{aligned}
(XY)A &= X(YA) && \text{by associativity} \\
&= X(AY) && \text{as } Y \text{ is in } V_A \\
&= (XA)Y && \text{by associativity} \\
&= (AX)Y && \text{as } X \text{ is in } V_A \\
&= A(XY) && \text{by associativity.}
\end{aligned}$$

Hence XY is in V_A and V_A is a subalgebra.
(ii) Let P be an invertible $n \times n$ matrix. Then

$$\begin{aligned}
V_{P^{-1}AP} &= \left\{B \in M_{nn} : B(P^{-1}AP) = (P^{-1}AP)B\right\} \\
&= \left\{B \in M_{nn} : \left(PBP^{-1}\right)A = A\left(PBP^{-1}\right)\right\} \\
&= \left\{B \in M_{nn} : PBP^{-1} \in V_A\right\} \\
&= \left\{B \in M_{nn} : B \in P^{-1}V_AP\right\} \\
&= P^{-1}V_AP.
\end{aligned}$$

(iii) Say that B is in V_A. Then

$$\begin{aligned}
BA^2 &= B(AA) \\
&= (BA)A && \text{by associativity} \\
&= (AB)A && \text{as } B \text{ is in } V_A \\
&= A(BA) && \text{by associativity} \\
&= A(AB) && \text{as } B \text{ is in } V_A \\
&= A^2B
\end{aligned}$$

and hence V_A is contained in V_{A^2}.

In general though V_A and V_{A^2} need not be equal. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{so that} \qquad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$V_{A^2} = M_{22} \qquad \text{but} \qquad V_A = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}\right\}.$$