Solution (#745) (i) Let $A = (a_{ij})$ be an upper triangular $n \times n$ matrix. Note that

$$A\mathbf{e}_1^T = a_{11}\mathbf{e}_1^T, \qquad A\mathbf{e}_2^T = a_{12}\mathbf{e}_1^T + a_{22}\mathbf{e}_2^T,, \qquad A\mathbf{e}_3^T = a_{13}\mathbf{e}_1^T + a_{23}\mathbf{e}_2^T + a_{33}\mathbf{e}_3^T$$

and in general

$$A\mathbf{e}_i^T \quad \text{is in} \quad \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_i^T \rangle$$

and

$$A\mathbf{e}_i^T \quad \text{is in} \quad \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_j^T \rangle \qquad \text{when } i \leqslant j.$$

So if we set $W_i = \langle \mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_i^T \rangle$ then W_i is A-invariant, $\dim W_i = i$ and W_i is contained in W_{i+1}.

(ii) Let B be an $n \times n$ matrix such that there are are B-invariant subspaces V_i of $\mathbb{R}_n$, for $i = 1, \ldots, n$, such that $\dim V_i = i$ and V_i is contained in V_{i+1} for $1 \leqslant i < n$. If there were an invertible matrix P such that $P^{-1}BP = A$ is upper triangular then we know from #744 (ii) that the B-invariant subspaces are PW where W is A-invariant. By the first part then we would see that the columns of P, which equal $P\mathbf{e}_i^T$ have the property that $P\mathbf{e}_i^T$ is in V_i and not in V_{i-1}.

So given such B-invariant subspaces V_i we will choose for each i a vector $\mathbf{v}_i$ in V_i that isn't in V_{i-1}. By #723 such vectors will form a basis and so $P = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \ldots \,|\, \mathbf{v}_n)$ is invertible. As the subspaces V_i are B-invariant then we have

$$B\mathbf{v}_1 = a_{11}\mathbf{v}_1, \qquad B\mathbf{v}_2 = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2, \qquad B\mathbf{v}_3 = a_{13}\mathbf{v}_1 + a_{23}\mathbf{v}_2 + a_{33}\mathbf{v}_3,$$

and so on. Hence

$$\begin{aligned} BP &= (B\mathbf{v}_1 \,|\, B\mathbf{v}_2 \,|\, \ldots \,|\, B\mathbf{v}_n) \\ &= (a_{11}\mathbf{v}_1 \,|\, a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 \,|\, \ldots \,|\, a_{13}\mathbf{v}_1 + a_{23}\mathbf{v}_2 + a_{33}\mathbf{v}_3) \\ &= (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \ldots \,|\, \mathbf{v}_n) \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22} & a_{23} & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = PA. \end{aligned}$$

Hence $P^{-1}BP = A$ as required.