Solution (#758) We may identify the space V of polynomials in x of degree 3 or less with $\mathbb{R}^4$ by $ax^3+bx^2+cx+d \longleftrightarrow (a,b,c,d)$.

(i) Let $W_1 = \{p(x) \in V : p(1) = 0\}$. Note that the zero polynomial is in W_1. If $p(x), q(x)$ are in W_1 and α, β are real scalars then

$$(\alpha p + \beta q)(1) = \alpha p(1) + \beta q(1) = \alpha 0 + \beta 0 = 0.$$

Hence W_1 is a subspace. For any $p(x)$ in W_1 we can write $p(x) = (x-1)(a+bx+cx^2)$ for some a, b, c. A basis for W_1 is then

$$(x-1), \qquad x(x-1), \qquad x^2(x-1),$$

and we have $\dim W_1 = 3$.

(ii) Let $W_2 = \{p(x) \in V : p(1) = p(2) = 0\}$. Note that the zero polynomial is in W_2. If $p(x), q(x)$ are in W_2 and α, β are real scalars then

$$\begin{aligned}(\alpha p + \beta q)(1) &= \alpha p(1) + \beta q(1) = \alpha 0 + \beta 0 = 0;\\ (\alpha p + \beta q)(2) &= \alpha p(2) + \beta q(2) = \alpha 0 + \beta 0 = 0.\end{aligned}$$

Hence W_2 is a subspace. For any $p(x)$ in W_2 we can write $p(x) = (x-1)(x-2)(a+bx)$ for some a, b. A basis for W_2 is then

$$(x-1)(x-2), \qquad x(x-1)(x-2),$$

and we have $\dim W_2 = 2$.

(iii) Let $W_3 = \{p(x) \in V : p(1) = p'(1) = 0\}$. Note that the zero polynomial is in W_3. If $p(x), q(x)$ are in W_3 and α, β are real scalars then

$$\begin{aligned}(\alpha p + \beta q)(1) &= \alpha p(1) + \beta q(1) = \alpha 0 + \beta 0 = 0;\\ (\alpha p + \beta q)'(1) &= \alpha p'(1) + \beta q'(1) = \alpha 0 + \beta 0 = 0.\end{aligned}$$

Hence W_3 is a subspace. For any $p(x)$ in W_3 we can write $p(x) = (x-1)^2(a+bx)$ for some a, b as the conditions $p(1) = p'(1) = 0$ are equivalent to $p(x)$ having $x = 1$ as a double root. A basis for W_3 is then

$$(x-1)^2, \qquad (x-1)^2 x,$$

and we have $\dim W_3 = 2$.

(iv) Let $W_4 = \{p(x) \in V : xp'(x) = 2p(x)\}$. Note that the zero polynomial is in W_4. If $p(x), q(x)$ are in W_4 and α, β are real scalars then

$$x(\alpha p(x) + \beta q(x))' = \alpha x p'(x) + \beta x q'(x) = \alpha(2p(x)) + \beta x(2q(x)) = 2(\alpha p(x) + \beta q(x)).$$

Hence W_4 is a subspace. If $p(x) = a + bx + cx^2 + dx^3$ is in W_4 then

$$x\left(b + 2cx + 3dx^2\right) = 2\left(a + bx + cx^2 + dx^3\right),$$

so comparing coefficients we find $0 = a = b = d$ with no condition on c. Hence

$$W_4 = \langle x^2 \rangle,$$

so that a basis for W_4 is x^2 and $\dim W_4 = 1$.

Remark: Alternatively we could have done each part as we did (iv) above. For example with W_2, the conditions $p(1) = p(2) = 0$ correspond to linear homogeneous conditions in the coefficients a, b, c, d. In this case

$$a + b + c + d = 0, \qquad a + 2b + 4c + 8d = 0.$$

Thus W_2 is the solution space of these two equations and hence a subspace. We could have found a basis for the solutions space by row-reducing in the usual way and assigning parameters.