Solution (#767) Let A be an $m \times n$ matrix so that $A^T A$ is $n \times n$. Then by the rank-nullity theorem we have

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n = \operatorname{rank}(A^T A) + \operatorname{nullity}(A^T A).$$

If $A\mathbf{v} = \mathbf{0}$ then we have $A^T A\mathbf{v} = \mathbf{0}$. Conversely if $A^T A\mathbf{v} = \mathbf{0}$ then

$$0 = \mathbf{v}^T A^T A\mathbf{v} = (A\mathbf{v})^T (A\mathbf{v}) = |A\mathbf{v}|^2$$

and so $A\mathbf{v} = \mathbf{0}$. Thus we have $\operatorname{Null}(A) = \operatorname{Null}(A^T A)$ and in particular

$$\operatorname{nullity}(A^T A) = \operatorname{nullity}(A)$$

and consequently $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$ by the rank-nullity theorem.