Solution (#768) Let A be an $m \times n$ matrix of rank r and

$$V = \{X \in M_{np} : AX = 0\}.$$

If X_1 and X_2 are in V and α_1, α_2 are real scalars then

$$A(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 A X_1 + \alpha_2 A X_2 = \alpha_1 0_{mp} + \alpha_2 0_{mp} = 0_{mp},$$

and so $\alpha_1 X_1 + \alpha_2 X_2$ lies in V. Clearly also 0_{np} lies in V, and so V is a subspace of M_{np}.

If X is in V and $\mathbf{c}_1, \ldots, \mathbf{c}_p$ are the columns of X then

$$AX = (A\mathbf{c}_1 | \cdots | A\mathbf{c}_p) = 0_{mp}$$

if and only if each $\mathbf{c}_i$ is in the null space of the map μ_A from $\mathbb{R}_n$ to $\mathbb{R}_m$. By the rank-nullity theorem we know that this null space has dimension $n - r$. Thus a matrix X in V is made up of column vectors from the null space of μ_A and these p columns may be independently chosen. It follows that

$$\dim V = p(n - r).$$