Solution (#775) Let A, B, C be $m \times n$ matrices.

(i) A is equivalent to A as $A = I_m A I_n$.

(ii) Say A is equivalent to B. Then there exist invertible P and Q such that $PAQ = B$. Then

$$P^{-1} B Q^{-1} = A$$

and hence B is equivalent to A.

(iii) Say A is equivalent to B, and B is equivalent to C. Then there are invertible matrices P_1, P_2, Q_1, Q_2 such that

$$A = P_1 B Q_1 \qquad \text{and} \qquad B = P_2 C Q_2.$$

We then have

$$A = P_1 B Q_1 = P_1 (P_2 C Q_2) Q_1 = (P_1 P_2) C (Q_2 Q_1),$$

noting that P_1P_2 and Q_1Q_2 are invertible as P_1, P_2, Q_1, Q_2 are. In particular A and B are equivalent.

(iv) Say that A and B are equivalent and that $A = PBQ$ where P and Q are invertible. Then we have

$$\begin{aligned}
\operatorname{rank}(PBQ) &= \operatorname{rank}(BQ) \qquad \text{[by Proposition 3.88(d)]} \\
&= \operatorname{rank}((BQ)^T) \qquad \text{[as row rank equals column rank]} \\
&= \operatorname{rank}(Q^T B^T) \qquad \text{[by the product rule for transposes]} \\
&= \operatorname{rank}(B^T) \qquad \text{[as } Q^T \text{ is invertible and Proposition 3.88(d)]} \\
&= \operatorname{rank}(B) \qquad \text{[as row rank equals column rank].}
\end{aligned}$$

Conversely say that A and B have the same rank r. We know by #691 that there are invertible matrices P_1, P_2, Q_1, Q_2 such that

$$P_1 A Q_1 = \begin{pmatrix} I_r & 0_{r(n-r)} \\ 0_{(m-r)r} & 0_{(m-r)(n-r)} \end{pmatrix} = P_2 B Q_2.$$

Hence by part (iii) A and B are equivalent.