Solution (#778) Let A be an $m \times n$ matrix of row rank r and let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ denote the rows of A. Recall from #711 that r is the largest number such that A has r independent rows.

Let Q be an $n \times n$ invertible matrix. The rows of AQ are $\mathbf{r}_1 Q, \ldots, \mathbf{r}_m Q$. So if there is a linear dependency in the rows of A

$$\alpha_1 \mathbf{r}_1 + \cdots + \alpha_m \mathbf{r}_m = \mathbf{0}$$

then there is a linear dependency in the rows of AQ

$$\alpha_1 (\mathbf{r}_1 Q) + \cdots + \alpha_m (\mathbf{r}_m Q) = (\alpha_1 \mathbf{r}_1 + \cdots + \alpha_m \mathbf{r}_m)\, Q = \mathbf{0}Q = \mathbf{0}.$$

It follows that the row rank of AQ is at least that of the row rank of A. But as Q^{-1} is also invertible it follows that the row rank of AQ in fact equals the row rank of A.

Let s denote the column rank of A and let P be an $m \times m$ invertible matrix. Now, as any linear dependency in the columns of A is equivalent to a linear dependency in the rows of A^T, we can see that s is the largest number such that A has s independent columns. As above, any dependency in the columns of A corresponds to a linear dependency in the columns of PA and thus the column rank of A equals the column rank of A.

We also recall at this point that the rowspace of PA equals the rowspace of A (Proposition 3.88(d)) and likewise that the column space of AQ equals the column space of A.

By #691 there is an invertible $m \times m$ matrix P and invertible $n \times n$ matrix Q such that

$$PAQ = \begin{pmatrix} I_r & 0_{r(n-r)} \\ 0_{(m-r)r} & 0_{(m-r)(n-r)} \end{pmatrix}.$$

The row rank and column rank of PAQ both equal r. So by the above we have

$$\begin{aligned} \text{row rank of } A &= \text{row rank of } PAQ \\ &= \text{column rank of } PAQ \\ &= \text{column rank of } A. \end{aligned}$$