Solution (#790) We have

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \qquad S_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Let $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{w} = (w_1, w_2)^T$. We then have

$$R_\theta(\mathbf{v}) = (v_1\cos\theta - v_2\sin\theta, v_1\sin\theta + v_2\cos\theta),$$

and so

$$\begin{aligned} R_\theta(\mathbf{v}) \cdot R_\theta(\mathbf{w}) &= (v_1\cos\theta - v_2\sin\theta, v_1\sin\theta + v_2\cos\theta) \cdot (w_1\cos\theta - w_2\sin\theta, w_1\sin\theta + w_2\cos\theta) \\ &= (v_1\cos\theta - v_2\sin\theta)(w_1\cos\theta - w_2\sin\theta) + (v_1\sin\theta + v_2\cos\theta)(w_1\sin\theta + w_2\cos\theta) \\ &= (v_1w_1 + v_2w_2)(\cos^2\theta + \sin^2\theta) + (v_1w_2 + v_2w_1)(-\sin\theta\cos\theta + \sin\theta\cos\theta) \\ &= v_1w_1 + v_2w_2 = \mathbf{v}\cdot\mathbf{w}. \end{aligned}$$

Similarly

$$S_\theta(\mathbf{v}) = (v_1\cos 2\theta + v_2\sin 2\theta, v_1\sin 2\theta - v_2\cos 2\theta),$$

and

$$\begin{aligned} S_\theta(\mathbf{v}) \cdot S_\theta(\mathbf{w}) &= (v_1\cos 2\theta + v_2\sin 2\theta, v_1\sin 2\theta - v_2\cos 2\theta) \cdot (w_1\cos 2\theta + w_2\sin 2\theta, w_1\sin 2\theta - w_2\cos 2\theta) \\ &= (v_1\cos 2\theta + v_2\sin 2\theta)(w_1\cos 2\theta + w_2\sin 2\theta) + (v_1\sin 2\theta - v_2\cos 2\theta)(w_1\sin 2\theta - w_2\cos 2\theta) \\ &= (v_1w_1 + v_2w_2)(\cos^2 2\theta + \sin^2 2\theta) + (v_1w_2 + v_2w_1)(\sin 2\theta\cos 2\theta - \sin 2\theta\cos 2\theta) \\ &= v_1w_1 + v_2w_2 = \mathbf{v}\cdot\mathbf{w}. \end{aligned}$$