Solution (#792) Let A be an $n \times n$ matrix.

(i) Suppose that A is invertible. A line in $\mathbb{R}^n$ can be parametrized as $\mathbf{r}(\lambda) = \mathbf{p} + \lambda \mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$. Its image is parametrized as

$$A\mathbf{r}(\lambda) = A(\mathbf{p} + \lambda \mathbf{v}) = A\mathbf{p} + \lambda A\mathbf{v}.$$

As A is invertible then $A\mathbf{v} \neq \mathbf{0}$ (Proposition 198) and so the image is still a line.

(ii) Say now that A is singular. Arguing as before we have image points

$$A\mathbf{r}(\lambda) = A\mathbf{p} + \lambda A\mathbf{v}.$$

If $A\mathbf{v} = \mathbf{0}$ then the image is the point $A\mathbf{p}$ and if $A\mathbf{v} \neq \mathbf{0}$ then the image is still a line as in the previous part.

(iii) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

which is singular. The line $y = x$ is mapped to the line $y = 2x$. The line $x + 2y = 0$ is entirely mapped to the origin.