Solution (#794) We have

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}; \qquad A_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$S_1 = \left\{(x,y)^T : x+y=1\right\}; \qquad S_2 = \left\{(x,y)^T : x,y>0\right\}; \qquad S_3 = \left\{(x,y)^T : x^2+y^2=1\right\}.$$

(i) and (iii): A general point of S_1 can be written $(t,1-t)^T$. Then

$$A_1 \begin{pmatrix} t \\ 1-t \end{pmatrix} = \begin{pmatrix} 1 \\ 2-2t \end{pmatrix}.$$

As t varies then this image vector parametrizes the entire line $x=1$. Similarly

$$A_3 \begin{pmatrix} t \\ 1-t \end{pmatrix} = \begin{pmatrix} 2t-1 \\ 1 \end{pmatrix}.$$

As t varies then this image vector parametrizes the entire line $y=1$.

(ii) and (iv): The image $A_1(S_2)$ consists of the vectors

$$(u+v,2v)^T = u\,(1,0)^T + v\,(1,2)^T \qquad \text{where} \qquad u,v>0.$$

These are all the vectors that lie between the positive x-axis and the half-line of $y=2x$ in the positive quadrant. Similarly $A_3(S_2)$ consists of the vectors

$$(u-v,u+v)^T = u\,(1,1)^T + v\,(-1,1)^T \qquad \text{where} \qquad u,v>0.$$

These are all the vectors the lie between the half-line of $y=x$ that lies in the first quadrant and the half-line of $x+y=0$ that lies in the second quadrant.

(v) A general point of S_3 has the form $(\cos\theta,\sin\theta)^T$ where $0 \leqslant \theta < 2\pi$. We have

$$A_3 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos(\theta+\pi/4) \\ \sin(\theta+\pi/4) \end{pmatrix}.$$

As θ varies these points map out the circle with centre the origin and radius $\sqrt{2}$.