Solution (#795) We have

$$A_2 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

and

$$S_1 = \left\{(x,y)^T : x+y=1\right\}; \qquad S_2 = \left\{(x,y)^T : x,y>0\right\}; \qquad S_3 = \left\{(x,y)^T : x^2+y^2=1\right\}.$$

Note that

$$A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+y \\ -2x+2y \end{pmatrix} = (y-x)\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

As x and y this vector gives all scalar multiples of $(1,2)^T$, which comprise the line $y=2x$. Note also that

$$(y-x)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} k \\ 2k \end{pmatrix} \iff y-x=k \iff y=x+k.$$

As the point $((1-k)/2,(k+1)/2)^T$ lies on the line S_1, with equation $x+y=1$, and maps to the point $(k,2k)^T$ then the image $A_2(S_1)$ is the entire line $y=2x$. Likewise, as the point

$$(|k|+1, |k|+k+1)$$

lies in the first quadrant S_2 and maps to the point $(k,2k)$ then the image $A_2(S_2)$ is the entire line $y=2x$. Finally A_2 maps a general point $(\cos\theta,\sin\theta)$ of S_3 to maps to

$$(\sin\theta-\cos\theta, 2\sin\theta-2\cos\theta) = \sqrt{2}\sin(\theta-\pi/4)(1,2)^T.$$

As θ varies we see that this gives all the point $(k,2k)$ where $-\sqrt{2}\leqslant k\leqslant\sqrt{2}$.

Finally

$$(A_2)^2 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1-2 & -1+2 \\ 2-4 & -2+4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix},$$

and

$$A_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{matrix} -x+y=x \\ -2x+2y=y \end{matrix} \iff y=2x.$$