Solution (#820) We shall prove this by induction on the columns of A. If A is 1×1 then by Definition 3.146 we have

$$\det \operatorname{diag}(A,B) = \det \begin{pmatrix} a_{11} & 0 \\ 0 & B \end{pmatrix} = a_{11} \det B = \det A \det B.$$

Suppose as an inductive hypothesis that the result $\det \operatorname{diag}(A,B) = \det A \det B$ applies for all matrices B and all $(k-1)\times(k-1)$ matrices A. Say now that A is a $k\times k$ matrix and B is $n\times n$, and set $M = \operatorname{diag}(A,B)$. By Definition 3.146 we have

$$\begin{aligned} \det \operatorname{diag}(A,B) &= [M]_{11} C_{11}(M) + [M]_{21} C_{21}(M) + \cdots + [M]_{k+n,1} C_{k+n,1}(M) \\ &= [A]_{11} C_{11}(M) + [A]_{21} C_{21}(M) + \cdots + [A]_{k1} C_{k1}(M), \end{aligned}$$

once we consider what the entries of the first column are. For $1 \leqslant i \leqslant k$ note, by our hypothesis, that

$$\begin{aligned} C_{i1}(M) &= (-1)^{i+1} \det \operatorname{diag}(A_{i1}, B) \\ &= (-1)^{i+1} \det A_{i1} \det B \\ &= C_{i1}(A) \det B. \end{aligned}$$

Thus we have

$$\begin{aligned} \det \operatorname{diag}(A,B) &= [A]_{11} C_{11}(A) \det B + \cdots + [A]_{k1} C_{k1}(A) \det B \\ &= \{[A]_{11} C_{11}(A) + \cdots + [A]_{k1} C_{k1}(A)\} \det B \\ &= \det A \det B, \end{aligned}$$

and the result follows by induction.