Solution (#829) Let z_1, z_2, z_3, p, q be complex numbers and define

$$\Delta = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}. \qquad D = \begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{p} & 1 \\ q & \bar{q} & 1 \end{vmatrix}. \qquad \delta = \begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{q} & 1 \\ q & \bar{p} & 1 \end{vmatrix}.$$

(i) Supposing z_1, z_2, z_3 to be distinct, then

$$\begin{aligned} z_1,\ z_2,\ z_3 \text{ are collinear} &\iff \frac{z_3 - z_1}{z_2 - z_1} \quad \text{is real} \\ &\iff \frac{z_3 - z_1}{z_2 - z_1} = \frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \\ &\iff (\bar{z}_3 - \bar{z}_1)(z_2 - z_1) - (z_3 - z_1)(\bar{z}_2 - \bar{z}_1) = 0 \\ &\iff \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 & 0 \\ z_3 - z_1 & \bar{z}_3 - \bar{z}_1 & 0 \end{vmatrix} = 0 \\ &\iff \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \\ &\iff \Delta = 0. \end{aligned}$$

If any of the points z_1, z_2, z_3 coincide then $\Delta = 0$ as two rows are equal.

(ii) Hence z, p, q are collinear if and only if $D = 0$. So for given distinct p, q then $D = 0$ is the equation of the line connecting p and q.

(iii) Note that $\delta = 0$ if and only if

$$\begin{aligned} & \begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{q} & 1 \\ q & \bar{p} & 1 \end{vmatrix} = 0 \\ \iff & \begin{vmatrix} z & \bar{z} & 1 \\ p + q & \bar{p} + \bar{q} & 2 \\ (1-i)p + (1+i)q & (1+i)\bar{p} + (1-i)\bar{q} & 2 \end{vmatrix} = 0 \\ \iff & \begin{vmatrix} z & \bar{z} & 1 \\ \frac{p+q}{2} & \overline{\left(\frac{p+q}{2}\right)} & 1 \\ \frac{(1-i)p+(1+i)q}{2} & \overline{\left(\frac{(1-i)p+(1+i)q}{2}\right)} & 1 \end{vmatrix} = 0. \end{aligned}$$

Hence $\delta = 0$ represents the line passing through

$$\frac{p+q}{2} \quad \text{and} \quad \frac{(1-i)p + (1+i)q}{2} = \frac{p+q}{2} + i\left(\frac{q-p}{2}\right).$$

The point $(p+q)/2$ is the midpoint of the line segment connecting p and q. The vector $q-p$ is parallel to this segment and so $i(q-p)$ is perpendicular to the segment. Hence $\delta = 0$ represents the perpendicular bisector of p and q.