Solution (#833) (i) Calculation of the first determinant for small values of $n \geqslant 1$ gives

$$x, \quad 0, \quad -x^2, \quad -x^3, \quad 0, \quad x^5, \quad x^6, \quad 0, \ldots \tag{10.27}$$

By expanding along the first column

$$a_n = \underbrace{\begin{vmatrix} x & x^2 & 0 & \cdots & 0 \\ 1 & x & x^2 & \cdots & 0 \\ 0 & 1 & x & \ddots & \vdots \\ \vdots & \vdots & \ddots & x & x^2 \\ 0 & 0 & \cdots & 1 & x \end{vmatrix}}_{n\times n} = x \underbrace{\begin{vmatrix} x & x^2 & 0 & \cdots & 0 \\ 1 & x & x^2 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & x & x^2 \\ 0 & \cdots & 0 & 1 & x \end{vmatrix}}_{(n-1)\times(n-1)} - \underbrace{\begin{vmatrix} x^2 & 0 & 0 & \cdots & 0 \\ 1 & x & x^2 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & x & x^2 \\ 0 & \cdots & 0 & 1 & x \end{vmatrix}}_{(n-1)\times(n-1)} = xa_{n-1} - x^2 a_{n-2},$$

by expanding the last determinant along the first row. The auxiliary equation $\lambda^2 - x\lambda + x^2 = 0$ has roots $\lambda = x\alpha$ and $\lambda = x/\alpha$ where $\alpha = \operatorname{cis}(\pi/3)$ is a cube root of -1. Hence

$$a_n = (A\alpha^n + B\alpha^{-n})x^n,$$

for some A, B. From this we can see that $a_{n+3} = -x^3 a_n$ and hence the sequence a_n does indeed continue as is suggested by the list in (10.27).

(ii) Calculation of the second determinant for small values of $n \geqslant 1$ gives

$$x^2, \quad 0, \quad 0, \quad x^8, \quad x^{10}, \quad 0, \quad 0, \quad x^{16}, \ldots$$

By expanding along the first row

$$b_n = \underbrace{\begin{vmatrix} x^2 & x^3 & 0 & 0 & \cdots & 0 \\ x & x^2 & x^3 & 0 & \cdots & 0 \\ 1 & x & x^2 & x^3 & \ddots & \vdots \\ 0 & 1 & x & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x^2 & x^3 \\ 0 & 0 & \cdots & 1 & x & x^2 \end{vmatrix}}_{n\times n} = x^2 \underbrace{\begin{vmatrix} x^2 & x^3 & 0 & 0 & \cdots & 0 \\ x & x^2 & x^3 & 0 & \cdots & 0 \\ 1 & x & x^2 & x^3 & \ddots & \vdots \\ 0 & 1 & x & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x^2 & x^3 \\ 0 & 0 & \cdots & 1 & x & x^2 \end{vmatrix}}_{(n-1)\times(n-1)} - x^3 \underbrace{\begin{vmatrix} x & x^3 & 0 & 0 & \cdots & 0 \\ 1 & x^2 & x^3 & 0 & \cdots & 0 \\ 0 & x & x^2 & x^3 & \ddots & \vdots \\ 0 & 1 & x & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x^2 & x^3 \\ 0 & 0 & \cdots & 1 & x & x^2 \end{vmatrix}}_{(n-1)\times(n-1)}$$

By the last determinant along the first row again and then down the first column we find

$$b_n = x^2 b_{n-1} - x^3(xb_{n-2} - x^3 b_{n-3}) = x^2 b_{n-1} - x^4 b_{n-2} + x^6 b_{n-3}.$$

So we have $b_n = x^2 b_{n-1} - x^4 b_{n-2} + x^6 b_{n-3}$. From this we can see that

$$\begin{aligned} b_{n+4} &= x^2 b_{n+3} - x^4 b_{n+2} + x^6 b_{n+1} \\ &= x^2 \left(x^2 b_{n+2} - x^4 b_{n+1} + x^6 b_n\right) - x^4 b_{n+2} + x^6 b_{n+1} \\ &= x^8 b_n. \end{aligned}$$

So the pattern repeats as we saw for small values: $b_n = e_n x^{2n}$ where e_n repeats the sequence $1, 0, 0, 1$ with period 4.

(iii) Denote the determinant b_n from (ii) now as $b_n(x)$ to stress its dependence on x. As $\det M = \det M^t$ for square matrices we then have that $c_n(x)$ equals

$$\begin{vmatrix} x & x^2 & x^3 & 0 & \cdots & 0 \\ 1 & x & x^2 & x^3 & \cdots & 0 \\ 0 & 1 & x & x^2 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x & x^2 \\ 0 & 0 & \cdots & 0 & 1 & x \end{vmatrix} = \begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ x^2 & x & 1 & 0 & \cdots & 0 \\ x^3 & x^2 & x & 1 & \ddots & \vdots \\ 0 & x^3 & x^2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x & 1 \\ 0 & 0 & \cdots & x^3 & x^2 & x \end{vmatrix} = x^{3n} \begin{vmatrix} x^{-2} & x^{-3} & 0 & 0 & \cdots & 0 \\ x^{-1} & x^{-2} & x^{-3} & 0 & \cdots & 0 \\ 1 & x^{-1} & x^{-2} & x^{-3} & \ddots & \vdots \\ 0 & 1 & x^{-1} & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x^{-2} & x^{-3} \\ 0 & 0 & \cdots & 1 & x^{-1} & x^{-2} \end{vmatrix}$$

which is $x^{3n} b_n\left(x^{-1}\right)$. So with e_n again denoting the repeating sequence $1, 0, 0, 1, 1, 0, 0, 1, \ldots$ we see

$$c_n = x^{3n} e_n \left(x^{-1}\right)^{2n} = e_n x^n.$$